# SELF-DUALITY IN THE CASE OF $SO(2n, F)$

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Abstract. The parabolically induced representations of special even-orthogonal groups over $p$-adic field are considered. The main result is a theorem on self-duality, which gives a condition on initial representations, if induced representation has a square integrable subquotient.

## 1. Introduction

The problem of construction of noncuspidal irreducible square integrable representations of classical $p$-adic groups was studied by M.Tadić in [T3]. He showed ([T3], Lemma 4.1) that among irreducible cuspidal representations of general linear groups only the self-dual play a role in the construction of irreducible noncuspidal square integrable representations of symplectic and odd-orthogonal groups.

In this paper, we show the same property for groups $SO(2n, F)$ (Theorem 6.1). In the second section, we review some notation and results from the representation theory of general linear groups. In the third section, we describe standard parabolics of $SO(2n, F)$. Some properties of induced representations of $SO(2n, F)$ are given in the fourth section. The fifth section exposes the Casselman square integrability criterion for $SO(2n, F)$. In the sixth section, a theorem on self-duality is stated and proved.

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## 2. Preliminaries

Fix a locally compact nonarchimedean field $F$ of characteristic different from 2. Let $G$ be a group of $F$-points of a connected reductive $F$-split group. Suppose that $G$ is reductive and split.

Fix a minimal parabolic subgroup $P_0 \subset G$ and a maximal split torus $A_0 \subset P_0$.

Let $P$ be a parabolic subgroup, containing $P_0$. We call such a group a standard parabolic subgroup. Let $U$ be the unipotent radical of $P$. Then, by [BZ], there exists a unique Levi subgroup $M$ in $P$ containing $A_0$.

Let $P$ be a standard parabolic subgroup of $G$, with Levi decomposition $P = MU$. For a smooth representation $\sigma$ of $M$, we denote by $i_{G,M}(\sigma)$ the parabolically induced representation of $G$ by $\sigma$ from $P$, and for a smooth representation $\pi$ of $G$, we denote by $r_{M,G}(\pi)$ the normalised Jacquet module of $\pi$ with respect to $P$.

For a smooth finite length representation $\pi$ we denote by $s.s.(\pi)$ the semi-simplified representation of $\pi$. The equivalence $s.s.(\pi_1) \cong s.s.(\pi_2)$ means that $\pi_1$ and $\pi_2$ have the same irreducible composition factors with the same multiplicities, and we write $\pi_1 = \pi_2$. We write $\pi_1 \cong \pi_2$ if we mean that $\pi_1$ and $\pi_2$ are actually equivalent.

Now we shall recall some results from [BZ] and [Z] of the representation theory of general linear groups.

For the group $GL(n, F)$, we fix the minimal parabolic subgroup which consists of all upper triangular matrices in $GL(n, F)$. The standard parabolic subgroups of $GL(n, F)$ can be parametrized by ordered partitions of $n$: for $\alpha = (n_1, \dots, n_k)$ there exists a standard parabolic subgroup (denote it in this section by $P_\alpha$) of $GL(n, F)$ whose Levi factor $M_\alpha$ is naturally isomorphic to $GL(n_1, F) \times \cdots \times GL(n_k, F)$.

Let $\pi_1, \pi_2$ be admissible representation of $GL(n_1, F), GL(n_2, F)$ resp., $n_1 + n_2 = n$. Define

$$\pi_1 \times \pi_2 = i_{GL(n,F),M(n_1+n_2)}(\pi_1 \otimes \pi_2).$$

Denote $\nu = |\det|$. We have the following criterion for irreducibility ([Z], Proposition1.11):

**Proposition 2.1.** *Let $\pi_i, i = 1, 2,$ be irreducible cuspidal representation of $GL(n_i, F)$.*

1. *If $\pi_1 \ncong \nu\pi_2$ and $\pi_2 \ncong \nu\pi_1$ (in particular if $n_1 \neq n_2$), then $\pi_1 \times \pi_2$ is irreducible.*
2. *Suppose that $n_1 = n_2$ and either $\pi_1 \cong \nu\pi_2$ or $\pi_2 \cong \nu\pi_1$. Then the representation $\pi_1 \times \pi_2$ has length 2.*

## 3. Parabolic induction for $SO(2n, F)$

The special orthogonal group $SO(2n, F)$, $n \geq 1$, is the group

$$SO(2n, F) = \{X \in SL(2n, F) \mid \ {}^{\tau}XX = I_{2n}\}.$$

Here ${}^{\tau}X$ denotes the transposed matrix of $X$ with respect to the second diagonal. For $n = 1$ we get

$$SO(2, F) = \left\{ \left[ \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right] \middle| \lambda \in F^{\times} \right\} \cong F^{\times}.$$

$SO(0, F)$ is defined to be the trivial group.

Denote by $A_0$ the maximal split torus in $SO(2n, F)$ which consists of all diagonal matrices in $SO(2n, F)$. Hence,

$$A_0 = \left\{ diag(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}) \middle| x_i \in F^{\times} \right\} \cong (F^{\times})^n.$$

Fix the minimal parabolic subgroup $P_0$ which consists of all upper triangular matrices in $SO(2n, F)$.

The root system is of type $D_n$; the simple roots are

$$\begin{array}{rcll} \alpha_i & = & e_i - e_{i+1}, & \text{for } 1 \leq i \leq n-1, \\ \alpha_n & = & e_{n-1} + e_n. & \end{array}$$

The set of simple roots is denoted by $\Delta$.

Let

$$s = \left[ \begin{array}{cccc} I & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I \end{array} \right] \in O(2n, F).$$

We use the same letter $s$ to denote the authomorphism of $SO(2n, F)$ defined by $s(g) = sgs^{-1}$.

Let $\theta = \Delta \setminus \{\alpha_i\}$, $i \in \{1, \ldots, n\}$, and let $P_\theta = M_\theta U_\theta$ be the maximal parabolic subgroup determined by $\theta$.

If $i \neq n-1$, then

$$M_\theta = \left\{ diag(g, h, \ {}^{\tau}g^{-1}) \mid g \in GL(i, F), \ h \in SO(2(n-i), F) \right\}.$$

In this case, we denote $M_\theta$ by $M_{(i)}$, and we have

$$M_{(i)} \cong GL(i, F) \times SO(2(n-i), F).$$

If $i = n-1$, then

$$M_\theta = s(M_{(n)}).$$

Now let $\theta = \Delta \setminus \{\alpha_{n-1}, \alpha_n\}$. Then

$M_\theta = \left\{ diag(g, h, \ ^\tau g^{-1}) \mid g \in GL(n-1, F), \ h \in SO(2, F) \cong GL(1, F) \right\}$,

so

$$\begin{aligned} M_\theta &\cong GL(n-1, F) \times SO(2, F), \\ M_\theta &\cong GL(n-1, F) \times GL(1, F), \end{aligned}$$

and we denote $M_\theta$ by $M_{(n-1)}$ or by $M_{(n-1,1)}$.

We shall now describe the set of standard parabolic subgroups of $SO(2n, F)$. Let $\alpha = (n_1, ..., n_k)$ be an ordered partition of non-negative integer $m \leq n$. Then there exists a standard parabolic subgroup, denote it by $P_\alpha = M_\alpha U_\alpha$, such that

$M_\alpha = \left\{ diag(g_1, ..., g_k, h, \ ^\tau g_k^{-1}, ..., \ ^\tau g_1^{-1}) \mid g_i \in GL(n_i, F), \ h \in SO(2(n-m), F) \right\}$.

Hence,

$$M_\alpha \cong GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times SO(2(n-m), F).$$

Mention that if $n_1 + \cdots + n_k = n - 1$, then

$M = \left\{ diag(g_1, ..., g_k, h, \ ^\tau g_k^{-1}, ..., \ ^\tau g_1^{-1}) \mid g_i \in GL(n_i, F), \ h \in SO(2, F) \cong GL(1, F) \right\}$,

so we may consider

$$M \cong GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times SO(2, F),$$

or

$$M \cong GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times GL(1, F).$$

Hence, we can assign

$$M \longmapsto \alpha = (n_1, ..., n_k),$$

or

$$M \longmapsto \alpha' = (n_1, ..., n_k, 1).$$

Besides the subgroups of type $P_\alpha = M_\alpha U_\alpha$, there is also another type of standard parabolic subgroups. They can be described as

$$M = s(M'),$$

where $M' = M_\alpha$, for some $\alpha = (n_1, ..., n_k)$, $n_1 + \cdots + n_k = n$.

Now, take smooth finite length representations $\pi$ of $GL(n, F)$ and $\sigma$ of $SO(2m, F)$. Let $P_{(n)} = M_{(n)} U_{(n)}$ be a standard parabolic subgroup of $G = SO(2(m+n), F)$. Hence, $M_{(n)} \cong GL(n, F) \times SO(2m, F)$, so $\pi \otimes \sigma$ can be taken as a representation of $M_{(n)}$. Define

$$\pi \rtimes \sigma = i_{M_{(n)}, G}(\pi \otimes \sigma).$$

Note that in the case $G = SO(2,F)$, the induction does nothing, since

$$M_{(0)} = M_{(1)} = SO(2,F) \cong GL(1,F),$$

and for a smooth representation $\pi$ of $GL(1,F)$, we have

$$\pi \rtimes 1 = \pi.$$

It follows from [BZ], Prop.2.3, that for smooth representations $\pi_1$ of $GL(n_1,F)$, $\pi_2$ of $GL(n_2,F)$ and $\sigma$ of $SO(2m,F)$ we have

$$\pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \times \pi_2) \rtimes \sigma.$$

Let $\sigma$ be a finite length smooth representation of $SO(2n,F)$. Let $\alpha = (n_1, ..., n_k)$ be an ordered partition of a non-negative integer $m \leq n$. Define

$$s_\alpha(\sigma) = r_{M_\alpha, SO(2n,F)}(\sigma).$$

## 4. Some properties of parabolically induced representations of $SO(2n,F)$

Let $G = SO(2n,F)$. For $m < n$, let $P = P_{(m)}$ be the standard parabolic subgroup with Levi factor $M \cong GL(m,F) \times SO(2(n-m),F)$. Then

$$s(P) = P, \quad s(M) = M, \quad s(U) = U.$$

The following lemma can be proved directly:

**Lemma 4.1.** *For a smooth finite length representation $\pi$ of $GL(m,F)$ and a smooth finite length representation $\sigma$ of $SO(2(n-m),F)$, $m < n$, we have*

$$s(\pi \rtimes \sigma) \cong \pi \rtimes s(\sigma).$$

**Proposition 4.2.** *Let $\pi$ be a smooth finite length representation of $GL(m,F)$ and $\sigma$ be a smooth finite length representation of $SO(2(n-m),F)$. Then*

$$\tilde{\pi} \rtimes \sigma = s^m(\pi \rtimes \sigma).$$

*Particularly,*

1. *If $m$ is even, then*

$$\tilde{\pi} \rtimes \sigma = \pi \rtimes \sigma;$$

2. *If $m < n$ is odd, then*

$$\tilde{\pi} \rtimes \sigma = \pi \rtimes s(\sigma);$$

3. *If $m = n$ is odd, then*

$$\tilde{\pi} \rtimes 1 = s(\pi \rtimes 1).$$

(Here $\tilde{\pi}$ denotes contragredient representation of $\pi$.)

*Proof.* Denote

$$j = s^n \begin{bmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{bmatrix} \in SO(2n, F).$$

Conjugation with $j$ gives

$$j(\pi \otimes \sigma) \cong s^m({}^t\pi^{-1} \otimes \sigma).$$

Since $j(M) = s^n(M) = s^m(M)$, the groups $j(P)$ and $s^m(P)$ are associated, so we have by [BDK]

$$\pi \rtimes \sigma = j(\pi \rtimes \sigma) = s^m(\tilde{\pi} \rtimes \sigma).$$

□

The following lemma is well-known.

**Lemma 4.3.** *Let $\rho$ be an irreducible cuspidal unitary representation of $GL(m, F)$ and let $\sigma$ be an irreducible cuspidal representation of $SO(2l, F)$, $l \neq 1$. Take $\alpha \in \mathbb{R}$. If $(\nu^\alpha \rho) \rtimes \sigma$ reduces, then $\rho \cong \tilde{\rho}$ and $\sigma \cong s^m(\sigma)$.*

*Proof.* Suppose first that $\alpha = 0$. The Frobenius reciprocity for $\rho \rtimes \sigma$ and $\rho \otimes \sigma$ gives

$$Hom_G(\rho \rtimes \sigma, \rho \rtimes \sigma) \cong Hom_M(r_{M,G} \circ i_{G,M}(\rho \otimes \sigma), \rho \otimes \sigma).$$

Now we have from the Geometric lemma [BZ]

$$s.s.(r_{M,G} \circ i_{G,M}(\rho \otimes \sigma)) = \begin{cases} \rho \otimes \sigma + \tilde{\rho} \otimes s^m(\sigma), & \text{for } m < n \text{ or } m \text{ even,} \\ \rho \otimes 1, & \text{for } m = n \text{ odd.} \end{cases}$$

If $\rho \rtimes \sigma$ is reducible, then $\dim_{\mathbb{C}} Hom_G(\rho \rtimes \sigma, \rho \rtimes \sigma) > 1$. It follows $\rho \cong \tilde{\rho}$, $\sigma \cong s^m(\sigma)$.

Now, suppose that $\alpha \neq 0$ and that $(\nu^\alpha \rho) \rtimes \sigma$ is reducible. It follows from Proposition 7.1.3. [C] that $(\nu^\alpha \rho) \rtimes \sigma$ has a square integrable subquotient. Therefore, $(\nu^\alpha \rho) \rtimes \sigma$ and $(\nu^{-\alpha} \rho) \rtimes \sigma$ have a common subquotient, so we get $\nu^\alpha \rho \otimes \sigma \cong \nu^{-\alpha} \rho \otimes \sigma$ or $\nu^\alpha \rho \otimes \sigma \cong \nu^\alpha \tilde{\rho} \otimes s^m(\sigma)$. The first equivalence implies $\alpha = 0$. Hence, we have $\nu^\alpha \rho \otimes \sigma \cong \nu^\alpha \tilde{\rho} \otimes s^m(\sigma)$. It follows $\rho \cong \tilde{\rho}$, $\sigma \cong s^m(\sigma)$. □

## 5. Square integrability criteria for $SO(2n, F)$

We shall state the criterion that follows from the Casselman square integrability criterion ([C], Theorem 6.5.1), and it is analogous to those from [T3] for $GSp(n, F)$.

Define

$$\beta_i = (\underbrace{1, \dots, 1}_{i \text{ times}}, \ 0, \dots, 0) \in \mathbb{R}^n, \quad i \leq n-2,$$

$$\beta_{n-1} = (1, \dots, 1, -1) \in \mathbb{R}^n,$$
$$\beta_n = (1, \dots, 1, 1) \in \mathbb{R}^n.$$

Let $\pi$ be an irreducible smooth representation of $G = SO(2n, F)$. Let $P = MU$ be a standard parabolic subgroup, minimal among all standard parabolic subgroups which satisfy

$$r_{M,G}(\pi) \neq 0.$$

Let $\rho$ be an irreducible subqotient of $r_{M,G}(\pi)$.

If $P = P_\alpha$, where $\alpha = (n_1, \dots, n_k)$ is a partition of $m \leq n$, then

$$\rho = \rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma,$$

where $\rho_i$ are irreducible cuspidal representations of $GL(n_i, F)$, and $\sigma$ is an irreducible cuspidal representation of $SO(2(n-m), F)$. If $P$ is not of that type, then

$$\rho = s(\rho_1 \otimes \cdots \otimes \rho_{k-1} \otimes \rho_k \otimes 1) = \rho_1 \otimes \cdots \otimes \rho_{k-1} \otimes s(\rho_k \otimes 1),$$

where $\rho_i$ are irreducible cuspidal representations of $GL(n_i, F)$.

We have $\rho_i = \nu^{e(\rho_i)} \rho_i^u$, where $e(\rho_i) \in \mathbb{R}$ and $\rho_i^u$ is unitarizable. Define

$$e_*(\rho) = (\underbrace{e(\rho_1), \dots, e(\rho_1)}_{n_1 \text{ times}}, \ \dots, \underbrace{e(\rho_k), \dots, e(\rho_k)}_{n_k \text{ times}}, \ \underbrace{0, \dots, 0}_{n-m \text{ times}}).$$

(This definition concerns $\rho = \rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma$ as well as $\rho = s(\rho_1 \otimes \cdots \otimes \rho_k \otimes 1)$.)

If $\pi$ is square integrable, then

$$\begin{aligned} (e_*(\rho), \beta_{n_1}) &> 0, \\ (e_*(\rho), \beta_{n_1+n_2}) &> 0, \\ &\vdots \\ (e_*(\rho), \beta_{m-n_k}) &> 0, \\ (e_*(\rho), \beta_m) &> 0. \end{aligned}$$

(Here $(\ ,\ )$ denotes the standard inner product on $\mathbb{R}^n$.)

Conversely, if all above inequalities hold for any $\alpha$ and $\sigma$ as above, then $\pi$ is square integrable.

The criteria implies

$$\pi \text{ is square integrable} \Leftrightarrow s(\pi) \text{ is square integrable},$$

but this equivalence can also be proved easily directly from the definition of square integrability.

## 6. A theorem on self-duality

**Theorem 6.1.** *Suppose that $\rho_1, \rho_2, \dots, \rho_k$ are irreducible cuspidal representations of $GL(n_1, F), \dots, GL(n_k, F)$, resp., and $\sigma$ is an irreducible cuspidal representation of $SO(2l, F)$, $l \neq 1$. If $\rho_1 \times \dots \times \rho_k \rtimes \sigma$ contains a square integrable subquotient, then $\rho_i^u \cong (\rho_i^u)^\sim$, for any $i = 1, 2, \dots ., k$.*

*Proof.* The proof paralels that used in chapter 4 of [T3].

Set $n_1 + \dots + n_k = m$, $m + l = n$. Denote

$$\begin{aligned} G &= SO(2n, F), \\ M &= M_{(n_1, \dots, n_k)}, \\ \rho &= \rho_1 \otimes \dots \otimes \rho_k \otimes \sigma. \end{aligned}$$

Then

$$\rho_1 \times \dots \times \rho_k \rtimes \sigma = i_{G,M}(\rho).$$

Let $\pi$ be an irreducible square integrable subquotient of $i_{G,M}(\rho)$. First we shall prove the lemma under the assumption that $\pi$ is a subrepresentation of $i_{G,M}(\rho)$, or, equivalently, that $\rho$ is a quotient of $r_{M,G}(\pi)$.

Fix any $i_0 \in \{1, \dots .k\}$. Set

$$\begin{aligned} Y_{i_0}^0 &= \{i \in \{1, \dots .k\} \mid \exists \alpha \in \mathbb{Z} \text{ such that } \rho_{i_0} \cong \nu^\alpha \rho_i\}, \\ Y_{i_0}^1 &= \{i \in \{1, \dots .k\} \mid \exists \alpha \in \mathbb{Z} \text{ such that } \tilde{\rho}_{i_0} \cong \nu^\alpha \rho_i\}, \\ Y_{i_0} &= Y_{i_0}^0 \cup Y_{i_0}^1, \\ Y_{i_0}^c &= \{1, \dots .k\} \backslash Y_{i_0}. \end{aligned}$$

Suppose that $\rho_{i_0}^u \ncong (\rho_{i_0}^u)^\sim$. It follows from Proposition 2.1 that for any $j_0, j_0' \in Y_{i_0}^0$, $j_1, j_1' \in Y_{i_0}^1$ and $j_c \in Y_{i_0}^c$ we have

$$\begin{array}{ll} \rho_{j_0} \times \tilde{\rho}_{j_0'} \cong \tilde{\rho}_{j_0'} \times \rho_{j_0}, & \rho_{j_1} \times \tilde{\rho}_{j_1'} \cong \tilde{\rho}_{j_1'} \times \rho_{j_1}, \\ \rho_{j_0} \times \rho_{j_1} \cong \rho_{j_1} \times \rho_{j_0}, & \tilde{\rho}_{j_0} \times \tilde{\rho}_{j_1} \cong \tilde{\rho}_{j_1} \times \tilde{\rho}_{j_0}, \\ \rho_{j_0} \times \rho_{j_c} \cong \rho_{j_c} \times \rho_{j_0}, & \tilde{\rho}_{j_0} \times \rho_{j_c} \cong \rho_{j_c} \times \tilde{\rho}_{j_0}, \\ \rho_{j_1} \times \rho_{j_c} \cong \rho_{j_c} \times \rho_{j_1}, & \tilde{\rho}_{j_1} \times \rho_{j_c} \cong \rho_{j_c} \times \tilde{\rho}_{j_1}. \end{array}$$

If $n-m>1$, then, by Lemma 4.3, $\rho_{j_0}\rtimes\sigma$ and $\rho_{j_1}\rtimes\sigma$ are irreducible. Now we get from Proposition 4.2

$$\begin{aligned}\rho_{j_0}\rtimes\sigma &\cong \tilde{\rho}_{j_0}\rtimes s^{n_{j_0}}(\sigma),\\ \rho_{j_1}\rtimes\sigma &\cong \tilde{\rho}_{j_1}\rtimes s^{n_{j_1}}(\sigma),\end{aligned}$$

for $n-m>1$, and

$$\begin{aligned}\rho_{j_0}\rtimes 1 &\cong s^{n_{j_0}}(\tilde{\rho}_{j_0}\rtimes 1),\\ \rho_{j_1}\rtimes 1 &\cong s^{n_{j_1}}(\tilde{\rho}_{j_1}\rtimes 1),\end{aligned}$$

for $n=m$. Write

$$\begin{aligned}Y^0_{i_0} &= \{a_1,\dots,a_{k_0}\}, \quad a_i<a_j \text{ for } i<j,\\ Y^1_{i_0} &= \{b_1,\dots,b_{k_1}\}, \quad b_i<b_j \text{ for } i<j,\\ Y^c_{i_0} &= \{d_1,\dots,d_{k_c}\}, \quad d_i<d_j \text{ for } i<j.\end{aligned}$$

If $n-m\geq 2$, then we can repeat the proof from [T3], since we have just slightly different relations, and square integrability criteria are the same.

Let $m=n$. Set $\alpha=n_{\beta_{k_1}}$. Then

$$\begin{aligned}&\rho_1\times\cdots\times\rho_k\rtimes 1\cong\\ \cong\ &\rho_{a_1}\times\cdots\times\rho_{a_{k_0}}\times\rho_{d_1}\times\cdots\times\rho_{d_{k_c}}\times\rho_{b_1}\times\cdots\times\rho_{b_{k_1}}\rtimes 1\\ \cong\ &\rho_{a_1}\times\cdots\times\rho_{a_{k_0}}\times\rho_{d_1}\times\cdots\times\rho_{d_{k_c}}\times\rho_{b_1}\times\cdots\times\rho_{b_{k_1-1}}\times s^{\alpha}(\tilde{\rho}_{b_{k_1}}\rtimes 1)\\ \cong\ &s^{\alpha}(\rho_{a_1}\times\cdots\times\rho_{a_{k_0}}\times\rho_{d_1}\times\cdots\times\rho_{d_{k_c}}\times\rho_{b_1}\times\cdots\times\rho_{b_{k_1-1}}\times\tilde{\rho}_{b_{k_1}}\rtimes 1)\\ \cong\ &s^{\alpha}(\rho_{a_1}\times\cdots\times\rho_{a_{k_0}}\times\rho_{d_1}\times\cdots\times\rho_{d_{k_c}}\times\tilde{\rho}_{b_{k_1}}\times\rho_{b_1}\times\cdots\times\rho_{b_{k_1-1}}\rtimes 1).\end{aligned}$$

We proceed in the same way, and finally we get

$$\rho_1\times\cdots\times\rho_k\rtimes 1\cong s^{\gamma}(\rho_{a_1}\times\cdots\times\rho_{a_{k_0}}\times\tilde{\rho}_{b_{k_1}}\times\cdots\times\tilde{\rho}_{b_1}\times\rho_{d_1}\times\cdots\times\rho_{d_{k_c}}\rtimes 1),$$

where $\gamma=0$ or $1$.

In the same manner, we obtain

$$\rho_1\times\cdots\times\rho_k\rtimes 1\cong s^{\delta}(\rho_{b_1}\times\cdots\times\rho_{b_{k_1}}\times\tilde{\rho}_{a_{k_0}}\times\cdots\times\tilde{\rho}_{a_1}\times\rho_{d_1}\times\cdots\times\rho_{d_{k_c}}\rtimes 1),$$

where $\delta=0$ or $1$.

By the Frobenius reciprocity, the representations

$$\begin{aligned}\rho' &= s^{\gamma}(\rho_{a_1}\otimes\cdots\otimes\rho_{a_{k_0}}\otimes\tilde{\rho}_{b_{k_1}}\otimes\cdots\otimes\tilde{\rho}_{b_1}\otimes\rho_{d_1}\otimes\cdots\otimes\rho_{d_{k_c}}\otimes 1),\\ \rho'' &= s^{\delta}(\rho_{b_1}\otimes\cdots\otimes\rho_{b_{k_1}}\otimes\tilde{\rho}_{a_{k_0}}\otimes\cdots\otimes\tilde{\rho}_{a_1}\otimes\rho_{d_1}\otimes\cdots\otimes\rho_{d_{k_c}}\otimes 1)\end{aligned}$$

are the quotients of corresponding Jacquet modules. Now $\rho_{a_1}\times\cdots\times\rho_{a_{k_0}}\times\rho_{b_1}\times\cdots\times\rho_{b_{k_1}}$ is representation of $GL(u,F)$, for some $u\leq n$. If $u\neq n-1$ then $(\beta_u,e_*(\rho'))=-(\beta_u,e_*(\rho''))$. If $u=n-1$, then

$$(\beta_{n-1},e_*(\rho'))+(\beta_n,e_*(\rho'))=-(\beta_{n-1},e_*(\rho''))-(\beta_n,e_*(\rho'')).$$

Anyway, this contradicts the assumption that $\pi$ is square integrable.

Generally, let $\pi$ be an irreducible subquotient of $i_{G,M}(\rho)$. By [C], Corollary 7.2.2, there exists $w \in W = N_G(M)/M$ such that $\pi$ is a subrepresentation of $i_{G,M}(w(\rho))$. Let $\rho = \rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma$ and $w(\rho) = \delta_1 \otimes \cdots \otimes \delta_k \otimes \tau$. We apply the first part of the proof on $w(\rho)$, and we get $\delta_i^u \cong (\delta_i^u)^{\sim}$, $i = 1, 2, \ldots., k$. By [G], the sequence $\delta_1, \ldots, \delta_k$ is, up to a permutation and taking a contragredient, the sequence $\rho_1, \ldots., \rho_k$. □

**Theorem 6.2.** *Suppose that $\rho_1, \rho_2, \ldots, \rho_k$ are irreducible cuspidal representations of $GL(n_1, F), \ldots, GL(n_k, F)$, resp., and $\sigma$ is an irreducible cuspidal representation of $SO(2l, F)$, $l \neq 1$, such that $\rho_1 \times \cdots \times \rho_k \rtimes \sigma$ contains a square integrable subquotient. Further, assume that for each unitary representation $\rho$, the number $\alpha$, discussed in Lemma 4.3., satisfies $2\alpha \in \mathbb{Z}$. Then $2e(\rho_i) \in \mathbb{Z}$, for any $i = 1, 2, \ldots., k$.*

*Proof.* The proof is analogous to that of Theorem 6.1. □

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