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DEGENERATE PRINCIPAL SERIES FOR EVEN-ORTHOGONAL GROUPS

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ABSTRACT. Let F be a p-adic field of characteristic 0 and $G = O(2n, F)$ (resp. $SO(2n, F)$). A maximal parabolic subgroup of G has the form $P = MU$, with Levi factor $M \cong GL(k, F) \times O(2(n-k), F)$ (resp. $M \cong GL(k, F) \times SO(2(n-k), F)$). We consider a one-dimensional representation of M of the form $\chi \circ det_k \otimes triv_{(n-k)}$, with χ a one-dimensional representation of $F^{\times}$; this may be extended trivially to get a representation of P. We consider representations of the form $\mathrm{Ind}_P^G(\chi \circ det_k \otimes triv_{(n-k)}) \otimes 1$. (Our results also work when $G = O(2n, F)$ and the inducing representation is $(\chi \circ det_k \otimes det_{(n-k)}) \otimes 1$, using $det_{(n-k)}$ to denote the nontrivial character of $O(2(n-k), F)$.) More generally, we allow Zelevinsky segment representations for the inducing representations.

In this paper, we study the reducibility of such representations. We determine the reducibility points, give Langlands data and Jacquet modules for each of the irreducible composition factors, and describe how they are arranged into composition series. For $O(2n, F)$, we use Jacquet module methods to obtain our results; the results for $SO(2n, F)$ are obtained via an analysis of restrictions to $SO(2n, F)$.

1. Introduction

Let F be a p-adic field with $char F = 0$.

The basic purpose of this paper is to study degenerate principal series for $O(2n, F)$ and $SO(2n, F)$ (though we work in a more general setting, what might be called generalized degenerate principal series). This paper completes the analysis of reducibility for degenerate principal series for classical p-adic groups; the corresponding results for $SL(n, F)$ (cf. [B-Z] and [Tad1]), $Sp(2n, F)$ and $SO(2n+1, F)$ (cf. [Jan3]) are already known. We remark that while a fair amount of work on degenerate principal series for $Sp(2n, F)$, $SO(2n+1, F)$ had been done prior to [Jan3] (cf. [Gus], [K-R], [Jan1], [Jan2], [Tad4]), relatively little has been done for $SO(2n, F)$ or $O(2n, F)$ (though [Jan2] contains some results for $SO(2n, F)$). There are also results on degenerate principal series available for other p-adic groups (e.g., see [Mu], [K-S], [Ch]). For real and complex groups, there is significantly more available on degenerate principal series; we refer the reader to [H-L] for a discussion of these cases.

In [Jan3] (generalized) degenerate principal series for $SO(2n+1, F)$, $Sp(2n, F)$ are analyzed. Structural similarities between the two families of groups allow them to be treated together. A careful study of Jacquet modules—made possible by the results from [Tad3]—allowed us to determine the number of irreducible subquotients

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and give their Langlands data. This, in turn, made it possible to reconstruct the composition series.

These structural similarities are shared by $O(2n, F)$ (but not $SO(2n, F)$). Thus, we first focus on (generalized) degenerate principal series for $O(2n, F)$. Now, there were two obstacles to including $O(2n, F)$ in [Jan3]: (1) the Jacquet module results in [Tad3] did not apply to $O(2n, F)$, and (2) the Langlands classification required the underlying algebraic group to be connected. Since then, the Jacquet module structures of [Tad3] have been extended to $O(2n, F)$ (cf. [Ban1]); the Langlands classification was extended (to quasi-split groups with abelian component group) in [B-J1]. (We also need multiplicity one for the Langlands classification; for $O(2n, F)$, this follows from [B-J2] or the argument for Lemma 3.4 in [Jan4].) Thus, generalized degenerate principal series for $O(2n, F)$ may be handled in exactly the same manner as in [Jan3].

We do our analysis for $SO(2n, F)$ by using [G-K] to study restrictions of representations from $O(2n, F)$ to $SO(2n, F)$. (This approach to the study of representations of non-connected groups has been used by [Gol2], [G-H], e.g., though they use information for the connected component to obtain information for the non-connected group.) The key tools for doing our analysis are Proposition 4.5 in [B-J1] and the results of section 2 [G-K]. If π is an irreducible representation of $O(2n, F)$, these allow us to obtain the Langlands data for the component(s) of $\pi|_{SO(2n,F)}$ from the Langlands data for π.

Let us now discuss the contents in greater detail. In the next section, we introduce notation and review some results which will be needed in the rest of the paper.

We begin by discussing the generalized degenerate principal series for $O(2n, F)$ considered in this paper. As in [B-Z], we let $\nu = |det|$ for general linear groups and use $\times$ to denote parabolic induction (cf. section 2 for a more detailed explanation). If ρ_0 is an irreducible, unitary, supercuspidal representation of $GL(r_0, F)$, then

$$\nu^{\frac{-k+1}{2}}\rho_0 \times \nu^{\frac{-k+1}{2}+1}\rho_0 \times \cdots \times \nu^{\frac{k-1}{2}}\rho_0$$

has a unique irreducible subrepresentation which we denote by $\zeta(\rho_0, k)$. We note that if $\rho_0 = 1_{F^\times}$, we have $\zeta(\rho_0, k) = triv_{GL(k,F)}$. As in [Tad2], we use $\rtimes$ to denote parabolic induction for classical groups (cf. section 2 for a more detailed explanation). Let ρ be an irreducible, unitary, supercuspidal representation of $GL(r, F)$ and σ an irreducible, supercuspidal representation of $O(2m, F)$ (or $Sp(2m, F)$, $SO(2m + 1, F)$, $SO(2m, F)$). (We note that such a σ is necessarily unitarizable.) We say (ρ, σ) satisfies (C0) if (1) $\rho \rtimes \sigma$ is reducible and (2) $\nu^x\rho \rtimes \sigma$ is irreducible for all $x \in \mathbb{R} \setminus \{0\}$. If (ρ, σ) satisfies (C0), then

$$\nu^{-\ell+1}\rho \times \nu^{-\ell+2}\rho \times \cdots \times \nu^{-1}\rho \times \rho \rtimes \sigma$$

has two irreducible subrepresentations which we denote $\zeta_1(\rho, \ell; \sigma)$ and $\zeta_2(\rho, \ell; \sigma)$. We note that if $\rho = 1_{F^\times}$ and $\sigma = 1_{O(0,F)}$ (with $O(0, F)$ the trivial group), then $\zeta_1(\rho, \ell; \sigma) = triv_{O(2\ell,F)}$ and $\zeta_2(\rho, \ell; \sigma) = det_{O(2\ell,F)}$.

In the third section, we discuss the generalized degenerate principal series $\nu^\alpha\zeta(\rho_0, k) \rtimes \zeta_i(\rho, \ell; \sigma)$, $\alpha \in \mathbb{R}$, for $O(2n, F)$. The case $\rho_0 \cong \rho$ is covered by Proposition 3.2 (for $k = 1$, $\ell \geq 1$), Proposition 3.3 (for $\ell = 0$, $k \geq 2$), and Theorem 3.4 (for $k \geq 2$, $\ell \geq 1$). The case $\rho_0 \not\cong \rho$ is covered by Theorem 3.5 and Remark 3.6. In particular, we determine for which values of α reducibility occurs. When there is reducibility, we identify the irreducible subquotients (by giving their Langlands data), describe their composition series structure, and give information on their Jacquet modules.

At this point, several remarks are in order. First, the arguments needed to obtain the results in section 3 are essentially the same as those used in [Jan3]. For this reason, we are rather brief in our discussion; we are content to summarize the results, with suitable references to [Jan3], and thereby avoid repeating long arguments which contain nothing new. Second, the roles of $\zeta_1(\rho,\ell;\sigma)$ and $\zeta_2(\rho,\ell;\sigma)$ are interchangeable; either can serve as the $\zeta_1(\rho,\ell;\sigma)$ in the results. Thus, section 3 may also be used to deal with degenerate principal series where the one-dimensional representation of the orthogonal group is *det* rather than *triv*. Finally, the results of section 3 apply equally well to $Sp(2n,F)$ and $SO(2n+1,F)$. However, for these groups, the results do not say anything about degenerate principal series. The difference lies in the conditions on (ρ,σ): For degenerate principal series we want $\rho = 1_{F^\times}$ and σ trivial (for the rank 0 classical group, i.e., the trivial group). For $O(2n,F)$, this means (ρ,σ) satisfies (C0); for $Sp(2n,F)$, $SO(2n+1,F)$, this requires that (ρ,σ) satisfies (C1/2), (C1), respectively (cf. [Jan3] for a more detailed discussion regarding $Sp(2n,F)$, $SO(2n+1,F)$).

Our analysis of generalized degenerate principal series for $SO(2n,F)$ is done by combining the results for $O(2n,F)$ (section 3) with results on the restriction of representations from $O(2n,F)$ to $SO(2n,F)$ (section 4). We note that our results on restriction from $O(2n,F)$ to $SO(2n,F)$ are built in part on the results of section 2 [G-K] and Proposition 4.5 [B-J1]. In particular, in combination these may be used to tell, from the Langlands data of an irreducible representation π of $O(2n,F)$, whether its restriction to $SO(2n,F)$ is reducible and determine the Langlands data of the components of the restriction.

Included in section 4 is a discussion of cuspidal reducibility. We recall what this means for $Sp(2n,F)$, $SO(2n+1,F)$. Suppose ρ is an irreducible, unitary, supercuspidal representation of $GL(r,F)$ and σ an irreducible supercuspidal representation of $Sp(2m,F)$ or $SO(2m+1,F)$. If $\rho \not\cong \tilde{\rho}$ ($\tilde{\rho}$ the contragredient of ρ), then $\nu^x\rho \rtimes \sigma$ is irreducible for all $x \in \mathbb{R}$; if $\rho \cong \tilde{\rho}$, there exists a unique $\alpha(\rho,\sigma) \geq 0$ such that $\nu^\alpha\rho \rtimes \sigma$ is reducible (cf. [Sil2], [Sil3]). Characterizations of the cuspidal reducibility $\alpha(\rho,\sigma)$ (based on certain conjectures) are given in [Mœ], [Zh]. (Also noteworthy are the results of [Sh1], [Sh2] in the generic case and the examples from [M-R] and [Re].) Using our study of restriction/induction between $SO(2n,F)$ and $O(2n,F)$, the results of [Mœ], [Zh] may be extended to $O(2n,F)$ (cf. Corollary 4.4), noting that [Mœ], [Zh] also deal with cuspidal reducbility for $SO(2n,F)$. This is obtained from Proposition 4.3, which relates the cuspidal reducibility for (ρ,σ) to that of (ρ,σ_0), where σ_0 is a component of the restriction of σ to $SO(2m,F)$.

In section 5, we deal with generalized degenerate principal series for $SO(2n,F)$. Here, we have two situations to consider. Suppose (ρ,σ) (σ an irreducible, supercuspidal representation of $O(2m,F)$) satisfies (C0). Let σ_0 be a component of the restriction of σ to $SO(2m,F)$. Then, either (1) (ρ,σ_0) satisfies (C0), or (2) $\nu^x\rho \rtimes \sigma_0$ is irreducible for all $x \in \mathbb{R}$ (cf. Proposition 4.3 for a precise characterization). In the first case (i.e., (ρ,σ_0) (C0)), the generalized degenerate principal series $\nu^\alpha\zeta(\rho_0) \rtimes \zeta_1(\rho,\ell;\sigma_0)$ (for $SO(2n,F)$) behave like the generalized degenerate principal series $\nu^\alpha\zeta(\rho_0) \rtimes \zeta_1(\rho,\ell;\sigma)$ (for $O(2n,F)$). In fact, the results are sufficiently similar such that we include their statements in section 3, although the proofs are in section 5. In the second case (i.e., $\nu^x\rho \rtimes \sigma_0$ irreducible for all $x \in \mathbb{R}$), the results are not as similar. In this case, the results on generalized degenerate principal series for $\rho_0 \cong \rho$ are given in Proposition 5.1 (for $k = 1$, $\ell \geq 1$), Proposition 5.2 (for $\ell = 0$, $k \geq 2$), and Theorem 5.3 (for $k \geq 2$, $\ell \geq 1$). The results for $\rho_0 \not\cong \rho$ are covered by Theorem 5.5 and Remark 5.6. Jacquet module information is included in Propositions 5.1 and 5.2. For Theorems 5.3 and 5.5, a brief discussion of how to calculate Jacquet modules is given in Remark 5.4. We note that this is the case which includes the actual degenerate principal series for $SO(2n, F)$.

We close this introduction with a sort of user's guide, to help easily find the appropriate results on degenerate principal series. For $O(2n, F)$, the results on degenerate principal series of the form $\mathrm{Ind}(|det_{GL(k,F)}|^x \otimes triv_{O(2\ell,F)})$ ($x \in \mathbb{R}$) may be found by taking $\rho = 1_{F^\times}$ and $\sigma = 1_{O(0,F)}$ ($O(0, F)$ is the trivial group) in Proposition 3.2 (for $k = 1$), Proposition 3.3 (for $\ell = 0$), or Theorem 3.4 (when $k \geq 2$ and $\ell \geq 1$). The results for degenerate principal series of the form $\mathrm{Ind}(|det_{GL(k,F)}|^x \cdot (\chi \circ det_{GL(k,F)}) \otimes triv_{O(2\ell,F)})$ (χ a unitary character of $F^\times$) may be found by taking $\rho_0 = \chi$, $\rho = 1_{F^\times}$, $\sigma = 1_{O(0,F)}$ in Theorem 3.5 (if χ is of order two) or Remark 3.6 (if order $\chi > 2$). To deal with degenerate principal series of the form $\mathrm{Ind}(|det_{GL(k,F)}|^x \cdot (\chi \circ det_{GL(k,F)}) \otimes det_{O(2\ell,F)})$, use the same results, but with the roles of $\zeta_1(\rho, \ell; \sigma)$ and $\zeta_2(\rho, \ell; \sigma)$ reversed.

For $SO(2n, F)$, the results on degenerate principal series of the form $\mathrm{Ind}(|det_{GL(k,F)}|^x \otimes triv_{SO(2\ell,F)})$ ($x \in \mathbb{R}$) may be found by taking $\rho = 1_{F^\times}$ and $\sigma = 1_{SO(0,F)}$ ($SO(0, F)$ is the trivial group) in Proposition 5.1 (for $k = 1$), Proposition 5.2 (for $\ell = 0$), or Theorem 5.3 (when $k \geq 2$ and $\ell \geq 1$). The results for degenerate principal series of the form $\mathrm{Ind}(|det_{GL(k,F)}|^x \cdot (\chi \circ det_{GL(k,F)}) \otimes triv_{SO(2\ell,F)})$ (χ a unitary character of $F^\times$) may be found by taking $\rho_0 = \chi$, $\rho = 1_{F^\times}$, $\sigma = 1_{SO(0,F)}$ in Theorem 5.5 (if χ is of order two) or Remark 5.6 (if order $\chi > 2$).

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2. Preliminaries

In this section, we introduce notation and state the Langlands classification for $SO(2n, F)$ and $O(2n, F)$. An explanation how such a form of the Langlands classification follows from the general results in [B-J1] can be found in the Appendix.

Let F be a p-adic field with $char F = 0$. We make use of results from [Gol1], [Gol2] in this paper, hence we need this assumption.

In most of this paper, we work with the components (irreducible subquotients) of a representation rather than with the actual composition series. Suppose that π_1, π_2 are smooth finite length representations. We write $\pi_1 = \pi_2$ if π_1 and π_2 have the same components with the same multiplicities. We write $\pi_1 \cong \pi_2$ if π_1 and π_2 are actually equivalent.

The special orthogonal group $SO(2n, F)$, $n \geq 1$, is the group

$$SO(2n, F) = \{X \in SL(2n, F) \mid {}^{\tau}XX = I_{2n}\}.$$

Here ${}^{\tau}X$ denotes the matrix of X transposed with respect to the second diagonal. For $n = 1$, we get

$$SO(2, F) = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \middle| \lambda \in F^\times \right\} \cong F^\times.$$

$SO(0,F)$ is defined to be the trivial group. The orthogonal group $O(2n,F)$, $n \geq 1$, is the group

$$O(2n,F) = \{X \in GL(2n,F) \mid {}^{\tau}XX = I_{2n}\}.$$

We have

$$O(2n,F) = SO(2n,F) \rtimes \{1,s\},$$

where

$$s = \begin{bmatrix} I & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I \end{bmatrix} \in O(2n,F)$$

and it acts on $SO(2n,F)$ by conjugation. We take $O(0,F)$ to be the trivial group.

In the group $SO(2n,F)$, fix the minimal parabolic subgroup $P_\emptyset$ consisting of all upper triangular matrices in $SO(2n,F)$ and the maximal split torus $A_\emptyset$ consisting of all diagonal matrices in $SO(2n,F)$.

Let M^0 be the standard Levi subgroup of $G^0 = SO(2n,F)$. We denote by i_{G^0,M^0} the functor of parabolic induction and by r_{M^0,G^0} the Jacquet functor (cf. [B-Z]). Let $G = O(2n,F)$. We use the notation i_{G,G^0} and $r_{G^0,G}$ for induction and restriction of representations.

Suppose that $\rho_1,\ldots,\rho_k$ are representations of $GL(n_1,F),\ldots,GL(n_k,F)$ and τ_0 a representation of $SO(2m,F)$. Let $G^0 = SO(2n,F)$, where $n = n_1+\cdots+n_k+m$. Let

$$M^0 = \left\{diag(g_1,...,g_k,h,{}^{\tau}g_k^{-1},...,{}^{\tau}g_1^{-1}) \mid g_i \in GL(n_i,F),\ h \in SO(2m,F)\right\}.$$

Then M^0 is a standard Levi subgroup of G^0 (cf. Appendix or [Ban2]). Following [B-Z], [Tad2], set

$$\rho_1 \times \cdots \times \rho_k \rtimes \tau_0 = i_{G^0,M^0}(\rho_1 \otimes \cdots \otimes \rho_k \otimes \tau_0).$$

If $m = 0$ and $n_k > 1$, then sM^0s is another standard Levi subgroup of G^0. Let 1_0 denote the trivial representation of $SO(0,F)$. We write $\rho_1 \otimes \cdots \otimes \rho_{k-1} \otimes s(\rho_k \otimes 1_0)$ for the corresponding representation of sM^0s. According to [Ban2],

$$\begin{aligned} s(\rho_1 \times \cdots \times \rho_k \rtimes 1_0) &= \rho_1 \times \cdots \times \rho_{k-1} \rtimes s(\rho_k \rtimes 1_0) \\ &= i_{G^0,sM^0s}(\rho_1 \otimes \cdots \otimes \rho_{k-1} \otimes s(\rho_k \otimes 1_0)). \end{aligned}$$

As in [B-Z], we set $\nu = |det|$ for general linear groups. Let ρ be an irreducible representation of $GL(n,F)$. We say that ρ is essentially square-integrable (resp., essentially tempered) if there exists $e(\rho) \in \mathbb{R}$ such that $\nu^{-e(\rho)}\rho$ is square-integrable (resp., tempered).

Now, we discuss the Langlands classification for $SO(2n,F)$(cf. Appendix). Let ρ_i, $i = 1,\ldots,k$, be irreducible essentially square-integrable representations of $GL(n_i,F)$ and τ_0 an irreducible tempered representation of $SO(2m,F)$. Suppose that $m \geq 1$ and $e(\rho_1) \leq \cdots \leq e(\rho_k) < 0$. Then the representation $\rho_1 \times \cdots \times \rho_k \rtimes \tau_0$ has a unique irreducible subrepresentation which we denote by $L(\rho_1,\ldots,\rho_k;\tau_0)$. (We note that this formulation—using weak inequalities with essentially square-integrable representations in lieu of strict inequalities with essentially tempered representations—follows easily from the fact that an irreducible tempered representation of $GL(n,F)$ has the form $\delta_1 \times \cdots \times \delta_j$ with δ_i irreducible and square-integrable.) If $m = 0$, then

$$sL(\rho_1,\ldots,\rho_k;1_0) \ncong L(\rho_1,\ldots,\rho_k;1_0).$$

We have $sL(\rho_1,\dots,\rho_k;1_0)=L(s(\rho_1\otimes\cdots\otimes\dots\rho_k\otimes 1_0))$. Both $\rho_1\otimes\cdots\otimes\rho_k\otimes 1_0$ and $s(\rho_1\otimes\cdots\otimes\rho_k\otimes 1_0)$ appear in (2) and (3) of Proposition 6.3, i.e., constitute Langlands data. Further, any Langlands datum in (2) or (3) of Proposition 6.3 may be written as either $\rho_1\otimes\cdots\otimes\rho_k\otimes 1_0$ or $s(\rho_1\otimes\cdots\otimes\rho_k\otimes 1_0)$ with $\rho_1,\dots,\rho_k$ as above. To simplify notation, etc., we write $sL(\rho_1,\dots,\rho_k;1_0)$ rather than $L(s(\rho_1\otimes\cdots\otimes\rho_k\otimes 1_0))$ in these cases.

At times, it will be convenient not to have to worry about listing $\rho_1,\dots,\rho_k$ in increasing order. So, if $\rho_1,\dots,\rho_k$ satisfy $e(\rho_i)<0$, then there is some permutation $\rho_{\sigma_1},\dots,\rho_{\sigma_k}$ which satisfies $e(\rho_{\sigma_1})\le\cdots\le e(\rho_{\sigma_k})<0$. Then, by $L(\rho_1,\dots,\rho_k;\tau_0)$ we mean $L(\rho_{\sigma_1},\dots,\rho_{\sigma_k};\tau_0)$.

Suppose that $\rho_1,\dots,\rho_k$ are representations of $GL(n_1,F),\dots,GL(n_k,F)$ and τ a representation of $O(2m,F)$. Let $G=O(2n,F)$, where $n=n_1+\cdots+n_k+m$. Let

$$M=\left\{diag(g_1,...,g_k,h,{}^\tau g_k^{-1},...,{}^\tau g_1^{-1})\mid g_i\in GL(n_i,F),\ h\in O(2m,F)\right\}.$$

Then M is a standard Levi subgroup of G (cf. Appendix or [B-J1]). Set

$$\rho_1\times\cdots\times\rho_k\rtimes\tau=i_{M,G}(\rho_1\otimes\cdots\otimes\rho_k\otimes\tau).$$

In the case $m=0$, we denote the trivial representation of $O(0,F)$ by 1.

Now, we give the Langlands classification for $O(2n,F)$ (cf. Appendix). Let ρ_i, $i=1,\dots,k$, be an irreducible essentially square-integrable representation of $GL(n_i,F)$ and τ an irreducible tempered representation of $O(2m,F)$. Suppose that $e(\rho_1)\le\cdots\le e(\rho_k)<0$. Then the representation $\rho_1\times\cdots\times\rho_k\rtimes\tau$ has a unique irreducible subrepresentation which we denote by $L(\rho_1,\dots,\rho_k;\tau)$.

Let ρ be an irreducible unitary supercuspidal representation of $GL(n,F)$ and $k\in\mathbb{Z}$, $k>0$. Then the representation

$$\nu^{\frac{-k+1}{2}}\rho\times\nu^{\frac{-k+1}{2}+1}\rho\times\cdots\times\nu^{\frac{k-1}{2}}\rho$$

has a unique irreducible subrepresentation which we denote by $\zeta(\rho,k)$ and a unique irreducible quotient which we denote by $\delta(\rho,k)$ (cf. [Zel]).

Suppose that σ is an irreducible supercuspidal representation of $SO(2m,F)$ (respectively, $O(2m,F)$) satisfying

(C0) $\rho\rtimes\sigma$ is reducible and $\nu^\alpha\rho\rtimes\sigma$ is irreducible for all $\alpha\in\mathbb{R}$ with $\alpha\neq 0$.

Then $\rho\rtimes\sigma$ is the direct sum of two irreducible tempered representations. We write

$$\rho\rtimes\sigma=T_1(\rho;\sigma)\oplus T_2(\rho;\sigma).$$

Let $i=1,2$ and $\ell>1$. By Jacquet module considerations, the representation

$$\nu^{-\ell+1}\rho\times\nu^{-\ell+2}\rho\times\cdots\nu^{-1}\rho\rtimes T_i(\rho;\sigma)$$

has a unique irreducible subrepresentation which we denote by $\zeta_i(\rho,\ell;\sigma)$ and a unique irreducible quotient which we denote by $\delta([\nu\rho,\nu^{\ell-1}\rho];T_i(\rho;\sigma))$ (it is square-integrable for $\ell>0$). For convenience, we also use the segment notation of Zelevinsky: let

$$[\nu^\beta\rho,\nu^{\beta+m}\rho]=\nu^\beta\rho,\nu^{\beta+1}\rho,\dots,\nu^{\beta+m}\rho.$$

Then, e.g., we have $\zeta_i(\rho,\ell;\sigma)=L([\nu^{-\ell+1}\rho,\nu^{-1}\rho];T_i(\rho;\sigma))$.

Let ρ be an irreducible supercuspidal representation of $GL(m,F)$, m odd. Suppose that $\rho\cong\tilde{\rho}$. Then $\rho\rtimes 1_0$ is an irreducible tempered representation of $SO(2m,F)$. By Jacquet module considerations, the representation

$$\nu^{-\ell+1}\rho\times\nu^{-\ell+2}\rho\times\cdots\nu^{-1}\rho\times\rho\rtimes 1_0$$

has a unique irreducible subrepresentation which we denote by $\zeta(\rho,\ell;1_0)$ and a unique irreducible quotient which we denote by $\delta([\nu\rho,\nu^{\ell-1}\rho];\rho\rtimes 1_0)$ (it is square-integrable for $\ell>0$). Similarly, if σ_0 is an irreducible admissible representation of $SO(2n,F)$ such that $s\sigma_0\not\cong\sigma_0$, then $\rho\rtimes\sigma_0$ is an irreducible tempered representation of $SO(2(m+n),F)$. By Jacquet module considerations, the representation

$$\nu^{-\ell+1}\rho\times\nu^{-\ell+2}\rho\times\cdots\nu^{-1}\rho\times\rho\rtimes\sigma_0$$

has a unique irreducible subrepresentation which we denote by $\zeta(\rho,\ell;\sigma_0)$ and a unique irreducible quotient which we denote by $\delta([\nu\rho,\nu^{\ell-1}\rho];\rho\rtimes\sigma_0)$ (it is square-integrable for $\ell>0$).

We introduce an additional piece of notation for Jacquet modules. Let $G=SO(2n,F)$ (respectively, $G=O(2n,F)$) and π a representation of G. We write

$$s_{(m)}\pi=r_{M,G}(\pi),$$

where M is the standard Levi subgroup of G isomorphic to $GL(m,F)\times SO(2(n-m),F)$ (respectively, $GL(m,F)\times O(2(n-m),F)$).

3. Degenerate principal series for $O(2n,F)$

In this section, we determine the reducibility for (generalized) degenerate principal series for $O(2n,F)$. Suppose (ρ,σ) satisfy (C0) (for $O(2(m+r),F)$). We analyze the reducibility of $\zeta(\rho_0,k)\rtimes\zeta_1(\rho,\ell;\sigma)$ below. If $\rho_0\cong\rho$, the results are given in Proposition 3.2 (for $k=1$), Proposition 3.3 (for $\ell=0$), and Theorem 3.4 (for $k\geq 2$, $\ell\geq 1$). If $\rho_0\not\cong\rho$, the results are given in Theorem 3.5. We note that the results consist of determining the reducibility points, the Langlands data of the irreducible subquotients which appear, the composition series structure, and certain information on Jacquet modules. As the arguments required are essentially the same as in [Jan3], we omit the (rather lengthy) details.

The results also apply to $SO(2n+1,F)$, $Sp(2n,F)$, and $SO(2n,F)$ when (ρ,σ) satisfies (C0) (where σ is an irreducible supercuspidal representation of the corresponding group). For $SO(2n+1,F)$, $Sp(2n,F)$, the proofs are the same as for $O(2n,F)$; for $SO(2n,F)$, the proofs are given in section 5. To allow this sort of generality in the results, we use $G(n,F)$ to denote any of these groups.

We start with a preliminary result. While there are strong reasons to believe that $\zeta_i(\rho,\ell;\sigma)$ is unitary, to the best of our knowledge this remains unknown. The following lemma serves as a substitute for the unitarity of $\zeta_i(\rho,\ell;\sigma)$. (A similar result should have been included in [Jan3].)

Lemma 3.1. *Suppose ρ, ρ_0 are irreducible unitary supercuspidal representations of $GL(m,F)$, $GL(m_0,F)$ and σ an irreducible supercuspidal representation of $G(r,F)$. Further, assume that (ρ,σ) satisfies (C0). Then, if $\pi=\zeta(\rho_0,k)\rtimes\zeta_i(\rho,\ell;\sigma)$ is reducible, it decomposes as the direct sum of two irreducible, inequivalent representations.*

Proof. Let $\hat{}$ denote the duality of [Aub] (also, cf. [S-S]); for $O(2n,F)$ see [Jan6]). Then, $\hat{\pi}=\delta(\rho_0,k)\rtimes\delta_i(\rho,\ell;\sigma)$. By [Gol1], [Gol2], e.g., we know that if this reduces, it is the direct sum of two inequivalent, irreducible subrepresentations. By Théorème 1.7 and Corollaire 3.9 [Aub], π must have two inequivalent, irreducible subquotients. It remains to show that π decomposes as a direct sum.

Suppose π is reducible. Let π_1 and π_2 denote the irreducible subquotients. We claim that to show that $\pi=\pi_1\oplus\pi_2$, it is enough to show that $s_{(km_0)}(\pi_i)$ contains

$\zeta(\rho_0,k)\otimes\zeta_i(\rho,\ell;\sigma)$. One can see from [Tad3] (also, cf. pp. 74 and 75, [Jan2]) that the two copies of $\zeta(\rho_0,k)\otimes\zeta_i(\rho,\ell;\sigma)$ in $s_{(km_0)}(\pi)$ are the only irreducible subquotients of $s_{(km_0)}(\pi)$ having unitary central character (all the rest have central character with the exponent having negative real part). Thus, if $s_{(km_0)}(\pi_i)$ has $\zeta(\rho_0,k)\otimes\zeta_i(\rho,\ell;\sigma)$ as an irreducible subquotient, it follows from central character considerations and Frobenius reciprocity that π_i is a subrepresentation of $\zeta(\rho_0,k)\rtimes\zeta_i(\rho,\ell;\sigma)$. The claim follows. Finally, to see that $s_{(km_0)}(\pi_i)$ contains $\zeta(\rho_0,k)\otimes\zeta_i(\rho,\ell;\sigma)$, we again look to the dual side, where we have $\hat{\pi}=\hat{\pi}_1\oplus\hat{\pi}_2$. By Frobenius reciprocity, $s_{(km_0)}(\hat{\pi}_i)$ contains $\delta(\rho_0,k)\otimes\delta_i(\rho,\ell;\sigma)$. By Théorème 1.7, [Aub], we can conclude $s_{(km_0)}(\pi_i)$ contains $\zeta(\rho_0,k)\otimes\zeta_i(\rho,\ell;\sigma)$, as needed. □

Proposition 3.2. *Let ρ be an irreducible unitary supercuspidal representation of $GL(m,F)$ and σ an irreducible supercuspidal representation of $G(r,F)$ with (ρ,σ) satisfying (C0). Let $\pi=\nu^{\alpha}\rho\rtimes\zeta_1(\rho,\ell;\sigma)$ with $\alpha\in\mathbb{R}$, $\ell\geq 1$. Then, π is reducible if and only if $\alpha\in\{\pm1,\pm\ell\}$. Suppose π is reducible. By contragredience, we may assume that $\alpha\leq 0$.*

(1) $\alpha=-1$, $\ell=1$
$\pi=\pi_1+\pi_2+\pi_3$ with

$$\pi_1=L(\nu^{-1}\rho;T_1(\rho;\sigma)),\quad \pi_2=\delta(\nu\rho;T_1(\rho;\sigma)),\quad \pi_3=L(\nu^{-\frac{1}{2}}\delta(\rho,2);\sigma).$$

In this case, π_1 is the unique irreducible subrepresentation, π_2 is the unique irreducible quotient, and π_3 is a subquotient. We have

$$s_{(m)}\pi_1=\nu^{-1}\rho\otimes T_1(\rho;\sigma),$$

$$s_{(m)}\pi_2=\nu\rho\otimes T_1(\rho;\sigma),$$

$$s_{(m)}\pi_3=\rho\otimes L(\nu^{-1}\rho;\sigma).$$

(2) $\alpha=-1$, $\ell>1$
$\pi=\pi_1+\pi_2$ with

$$\pi_1=L([\nu^{-\ell+1}\rho,\nu^{-1}\rho],\nu^{-1}\rho;T_1(\rho;\sigma)),\quad \pi_2=L([\nu^{-\ell+1}\rho,\nu^{-1}\rho];\delta(\nu\rho;T_1(\rho;\sigma))).$$

In this case, π_1 is the unique irreducible subrepresentation and π_2 is the unique irreducible quotient. We have

(a) $\ell=2$

$$s_{(m)}\pi_1=2\nu^{-1}\rho\otimes L(\nu^{-1}\rho;T_1(\rho;\sigma))+\nu^{-1}\rho\otimes L(\nu^{-\frac{1}{2}}\delta(\rho,2);\sigma),$$
$$s_{(m)}\pi_2=\nu^{-1}\rho\otimes\delta(\nu\rho;T_1(\rho;\sigma))+\nu\rho\otimes L(\nu^{-1}\rho;T_1(\rho;\sigma)).$$

(b) $\ell>2$.

$$\begin{aligned}s_{(m)}\pi_1&=\nu^{-\ell+1}\rho\otimes L([\nu^{-\ell+2}\rho,\nu^{-1}\rho],\nu^{-1}\rho;T_1(\rho;\sigma))\\&\quad+\nu^{-1}\rho\otimes L([\nu^{-\ell+1}\rho,\nu^{-1}\rho];T_1(\rho;\sigma)),\\ s_{(m)}\pi_2&=\nu^{-\ell+1}\rho\otimes L([\nu^{-\ell+2}\rho,\nu^{-1}\rho];\delta(\nu\rho;T_1(\rho;\sigma)))\\&\quad+\nu\rho\otimes L([\nu^{-\ell+1}\rho,\nu^{-1}\rho];T_1(\rho;\sigma)).\end{aligned}$$

(3) $\alpha=-\ell$, $\ell>1$.
$\pi=\pi_1+\pi_2$ with

$$\pi_1=L([\nu^{-\ell}\rho,\nu^{-1}\rho];T_1(\rho;\sigma)),\quad \pi_2=L(\nu^{-\ell+\frac{1}{2}}\delta(\rho,2),[\nu^{-\ell+2}\rho,\nu^{-1}\rho];T_1(\rho;\sigma)).$$

In this case, π_1 is the unique irreducible subrepresentation and π_2 is the unique irreducible quotient. We have

$$\begin{aligned} s_{(m)}\pi_1 &= \nu^{-\ell}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; T_1(\rho;\sigma)) \\ s_{(m)}\pi_2 &= \nu^{-\ell+1}\rho \otimes L(\nu^{-\ell}\rho, [\nu^{-\ell+2}\rho, \nu^{-1}\rho]; T_1(\rho;\sigma)) \\ &\quad + \nu^{\ell}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; T_1(\rho;\sigma)). \end{aligned}$$

Proof. For $O(2n,F)$ (and $SO(2n+1,F)$, $Sp(2n,F)$), this is proved as in Proposition 3.1, [Jan3] (also, cf. [Tad4]). □

Proposition 3.3. *Let ρ be an irreducible unitary supercuspidal representation of $GL(m,F)$ and σ an irreducible supercuspidal representation of $G(r,F)$ with (ρ,σ) satisfying (C0). Let $\pi = \nu^{\alpha}\zeta(\rho,k) \rtimes \sigma$ with $\alpha \in \mathbb{R}$, $k \geq 2$. Then π is reducible if and only if $\alpha \in \{\frac{-k+1}{2}, \frac{-k+3}{2}, \ldots, \frac{k-1}{2}\}$. Suppose π is reducible. By contragredience, we may assume that $\alpha \leq 0$. Write $\alpha = \frac{-k+1}{2} + j$ with $0 \leq j \leq \frac{k-1}{2}$.*

(1) $j = \frac{k-1}{2}$

$\pi = \pi_1 + \pi_2$ with

$$\pi_i = L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho]; T_i(\rho;\sigma))$$

for $i = 1,2$. In this case, $\pi = \pi_1 \oplus \pi_2$. We have

$$\begin{aligned} s_{(m)}\pi_i &= 2\nu^{\frac{-k+1}{2}}\rho \otimes L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+3}{2}}\rho, \nu^{-1}\rho]; T_i(\rho;\sigma)) \\ &\quad + \nu^{\frac{-k+1}{2}}\rho \otimes L(\nu^{-\frac{k}{2}+1}\delta(\rho,2), \nu^{-\frac{k}{2}+2}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma) \end{aligned}$$

for $i = 1,2$.

(2) $0 \leq j < \frac{k-1}{2}$

$\pi = \pi_1 + \pi_2 + \pi_3$ with

$$\pi_i = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; T_i(\rho;\sigma))$$

for $i = 1,2$ and

$$\pi_3 = L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \nu^{-j+\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma).$$

In this case, π_3 is the unique irreducible quotient and $\pi_1 \oplus \pi_2$ is a subrepresentation.

(a) $j = 0 = \frac{k-2}{2}$ $(k = 2)$,

$$s_{(m)}\pi_i = \nu^{-1}\rho \otimes T_i(\rho;\sigma)$$

for $i = 1,2$.

$$s_{(m)}\pi_3 = \rho \otimes L(\nu^{-1}\rho;\sigma).$$

(b) $j = 0$, $k > 2$,

$$s_{(m)}\pi_i = \nu^{-k+1}\rho \otimes L([\nu^{-k+2}\rho, \nu^{-1}\rho]; T_i(\rho;\sigma))$$

for $i = 1,2$.

$$\begin{aligned} s_{(m)}\pi_3 &= \nu^{-k+1}\rho \otimes L([\nu^{-k+2}\rho, \nu^{-2}\rho], \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma) \\ &\quad + \rho \otimes L([\nu^{-k+1}\rho, \nu^{-1}\rho]; \sigma). \end{aligned}$$

(c) $j = \frac{k-2}{2}$, $k \geq 4$ (*k even*),

$$\begin{aligned} s_{(m)}\pi_i &= \nu^{-\frac{k}{2}}\rho \otimes L([\nu^{-\frac{k}{2}+1}\rho, \nu^{-1}\rho], [\nu^{-\frac{k}{2}+1}\rho, \nu^{-1}\rho]; T_i(\rho;\sigma)) \\ &+\nu^{-\frac{k}{2}+1}\rho \otimes L([\nu^{-\frac{k}{2}}\rho, \nu^{-1}\rho], [\nu^{-\frac{k}{2}+2}\rho, \nu^{-1}\rho]; T_i(\rho;\sigma)) \end{aligned}$$

for $i = 1, 2$.

$$s_{(m)}\pi_3 = \nu^{-\frac{k}{2}+1}\rho \otimes L(\nu^{-\frac{k}{2}}\rho, \nu^{\frac{-k+3}{2}}\delta(\rho,2), \dots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma).$$

(d) $0 < j < \frac{k-2}{2}$,

$$\begin{aligned} s_{(m)}\pi_i &= \nu^{-k+j+1}\rho \otimes L([\nu^{-k+j+2}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; T_i(\rho;\sigma)) \\ &+\nu^{-j}\rho \otimes L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j+1}\rho, \nu^{-1}\rho]; T_i(\rho;\sigma)) \end{aligned}$$

for $i = 1, 2$.

$$\begin{aligned} s_{(m)}\pi_3 &= \nu^{-k+j+1}\rho \otimes L([\nu^{-k+j+2}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \dots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma) \\ &+\nu^{-j}\rho \otimes L([\nu^{-k+j+1}\rho, \nu^{-j-1}\rho], \nu^{-j+\frac{1}{2}}\delta(\rho,2), \dots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma). \end{aligned}$$

Proof. For $O(2n, F)$, this is essentially the same as Proposition 3.11, [Jan3]. (For $SO(2n+1, F)$, $Sp(2n, F)$, this is Proposition 3.11, [Jan3].) □

Theorem 3.4. *Let ρ be an irreducible unitary supercuspidal representation of $GL(m, F)$ and σ an irreducible supercuspidal representation of $G(r, F)$ with (ρ, σ) satisfying (C0). Let $\pi = \nu^\alpha\zeta(\rho, k) \rtimes \zeta_1(\rho, \ell; \sigma)$. Suppose $k \geq 2$ and $\ell \geq 1$ (the cases $k = 1$ and $\ell = 0$ are covered by Propositions 3.2 and 3.3 above). Then, π is reducible if and only if*

$$\begin{aligned} \alpha \in \quad & \{\pm(\ell + \tfrac{k-1}{2}), \pm(\ell + \tfrac{k-1}{2} - 1), \dots, \pm(\ell + \tfrac{-k+1}{2})\} \\ & \cup \{\{\tfrac{-k-1}{2}, \tfrac{-k-1}{2} + 1, \dots, \tfrac{k+1}{2}\} \setminus \{0 \text{ if } k = 2\ell - 1\}\}. \end{aligned}$$

(We note that these sets need not be disjoint.) Let S_1 denote the first set and S_2 the second. Suppose π is reducible. By contragredience, we may restrict our attention to the case $\alpha \leq 0$.

(1) $\alpha \notin S_2$.

In this case, we have $\pi = \pi_1 + \pi_2$, where

$$\pi_1 = L([\nu^{\alpha-\frac{k-1}{2}}\rho, \nu^{\alpha+\frac{k-1}{2}}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; T_1(\rho;\sigma)),$$

$$\begin{aligned} \pi_2 = L([&\nu^{\alpha-\frac{k-1}{2}}\rho, \nu^{-\ell-1}\rho], \\ &\nu^{-\ell+\frac{1}{2}}\delta(\rho,2), \nu^{-\ell+\frac{3}{2}}\delta(\rho,2), \dots, \nu^{\alpha+\frac{k}{2}}\delta(\rho,2), [\nu^{\alpha+\frac{k-1}{2}+2}\rho, \nu^{-1}\rho]; T_1(\rho;\sigma)). \end{aligned}$$

π_1 is the unique irreducible subrepresentation and π_2 is the unique irreducible quotient.

(2) $\alpha = \frac{-k-1}{2}$.

One component of π is the following:

$$\pi_1 = L([\nu^{-k}\rho, \nu^{-1}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; T_1(\rho;\sigma)).$$

The other components are described below.

(a) $\ell = 1$ *(so $k > \ell - 1$).*

In this case, there are two additional components:

$$\pi_2 = L([\nu^{-k}\rho, \nu^{-2}\rho]; \delta(\nu\rho; T_1(\rho;\sigma))),$$

and

$$\pi_3 = L([\nu^{-k}\rho, \nu^{-2}\rho], \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma).$$

π_1 is the unique irreducible subrepresentation, π_2 is the unique irreducible quotient, and π_3 is a subquotient.

(b) $k > \ell - 1 > 0$.

In this case, there are three additional components:

$$\pi_2 = L([\nu^{-k}\rho, \nu^{-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \delta(\nu\rho; T_1(\rho;\sigma))),$$

$$\pi_3 = L([\nu^{-k}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho,2), \nu^{-\ell+\frac{3}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{3}{2}}\delta(\rho,2); \delta(\nu\rho; T_1(\rho;\sigma))),$$

$$\pi_4 = L([\nu^{-k}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho,2), \nu^{-\ell+\frac{3}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma).$$

π_1 is the unique irreducible subrepresentation, π_3 is the unique irreducible quotient, and $\pi_2 \oplus \pi_4$ is a subquotient.

(c) $\ell - 1 = k$.

In this case, there is one additional component:

$$\pi_2 = L([\nu^{-k}\rho, \nu^{-2}\rho], [\nu^{-k}\rho, \nu^{-1}\rho]; \delta(\nu\rho; T_1(\rho;\sigma))).$$

π_1 is the unique irreducible subrepresentation and π_2 is the unique irreducible quotient.

(d) $\ell - 1 > k$.

In this case, there is one additional component:

$$\pi_2 = L([\nu^{-\ell+1}\rho, \nu^{-2}\rho], [\nu^{-k}\rho, \nu^{-1}\rho]; \delta(\nu\rho; T_1(\rho;\sigma))).$$

π_1 is the unique irreducible subrepresentation and π_2 is the unique irreducible quotient.

(3) $\alpha \in S_2$.

Write $\alpha = \frac{-k+1}{2} + j$, with $0 \le j \le \frac{k-1}{2}$. One component of π is π_1, where π_1 is defined as follows:

$$\pi_1 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \rtimes T_1(\rho;\sigma)).$$

The remaining components are described below, on a case by case basis.

(a) $k - j - 1 > j > \ell - 1$.

We have two additional components:

$$\begin{aligned}\pi_2 = L([&\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-\ell-1}\rho],\\ &\nu^{-\ell+\frac{1}{2}}\delta(\rho,2), \nu^{-\ell+\frac{3}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); T_2(\rho;\sigma)),\\ \pi_3 = L([&\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho],\\ &\nu^{-j-\frac{1}{2}}\delta(\rho,2), \nu^{-j+\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); T_2(\rho;\sigma)).\end{aligned}$$

π_3 is the unique irreducible quotient and $\pi_1 \oplus \pi_2$ is a subrepresentation.

(b) $k - j - 1 = j > \ell - 1$.

We have one additional component:

$$\begin{aligned}\pi_2 = L([&\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-\ell-1}\rho],\\ &\nu^{-\ell+\frac{1}{2}}\delta(\rho,2), \nu^{-\ell+\frac{3}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); T_2(\rho;\sigma)).\end{aligned}$$

In this case, $\pi = \pi_1 \oplus \pi_2$.

(c) $k - j - 1 > j = \ell - 1$.

We have one additional component:

$$\begin{aligned}\pi_2 = L([&\nu^{-k+j+1}\rho, \nu^{-\ell-1}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho],\\ &\nu^{-\ell+\frac{1}{2}}\delta(\rho,2), \nu^{-\ell+\frac{3}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); T_2(\rho;\sigma)).\end{aligned}$$

π_1 is the unique irreducible subrepresentation and π_2 is the unique irreducible quotient.

(d) $k-j-1>\ell-1>j$.

We have three additional components:

$$\pi_2 = L([\nu^{-k+j+1}\rho, \nu^{-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \nu^{-j+\frac{1}{2}}\delta(\rho,2), \dots, \nu^{-\frac{1}{2}}\delta(\rho,2); \delta(\nu\rho; T_1(\rho;\sigma))),$$

$$\pi_3 = L([\nu^{-k+j+1}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho,2), \nu^{-\ell+\frac{3}{2}}\delta(\rho,2), \dots, \nu^{-j-\frac{5}{2}}\delta(\rho,2), \nu^{-j-1}\delta(\rho,3), \nu^{-j}\delta(\rho,3), \dots, \nu^{-1}\delta(\rho,3); \delta(\nu\rho; T_1(\rho;\sigma))).$$

$$\pi_4 = L([\nu^{-k+j+1}\rho, \nu^{-\ell-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho,2), \nu^{-\ell+\frac{3}{2}}\delta(\rho,2), \dots, \nu^{-\frac{1}{2}}\delta(\rho,2); T_2(\rho;\sigma)).$$

π_1 is the unique irreducible subrepresentation, π_3 is the unique irreducible quotient, and $\pi_2 \oplus \pi_4$ is a subquotient.

(e) $k-j-1=\ell-1>j$.

We have one additional component:

$$\pi_2 = L([\nu^{-\ell+1}\rho, \nu^{-k+\ell-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-2}\rho], \nu^{-k+\ell-\frac{1}{2}}\delta(\rho,2), \nu^{-k+\ell+\frac{1}{2}}\delta(\rho,2), \dots, \nu^{-\frac{1}{2}}\delta(\rho,2); \delta(\nu\rho; T_1(\rho;\sigma))).$$

π_1 is the unique irreducible subrepresentation and π_2 is the unique irreducible quotient.

(f) $\ell-1>k-j-1>j$.

(i) *If $j=0$, the representation π_2 below is the only other component. In this case, π_1 is the unique irreducible subrepresentation and π_2 is the unique irreducible quotient.*

(ii) *If $j>0$, there are two additional components:*

$$\pi_2 = L([\nu^{-k+j+1}\rho, \nu^{-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \nu^{-j+\frac{1}{2}}\delta(\rho,2), \dots, \nu^{-\frac{1}{2}}\delta(\rho,2); \delta(\nu\rho; T_1(\rho;\sigma))),$$

$$\pi_3 = L([\nu^{-\ell+1}\rho, \nu^{-k+j-1}\rho], [\nu^{-j}\rho, \nu^{-2}\rho], \nu^{-k+j+\frac{1}{2}}\delta(\rho,2), \nu^{-k+j+\frac{3}{2}}\delta(\rho,2), \dots, \nu^{-\frac{1}{2}}\delta(\rho,2); \delta(\nu\rho; T_1(\rho;\sigma))).$$

In this case, π_2 is the unique irreducible quotient and $\pi_1 \oplus \pi_3$ is a subrepresentation.

(g) $\ell-1>k-j-1=j$.

We have one additional component:

$$\pi_2 = L([\nu^{-\ell+1}\rho, \nu^{\frac{-k-3}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-2}\rho], \nu^{-\frac{k}{2}}\delta(\rho,2), \nu^{-\frac{k}{2}+1}\delta(\rho,2), \dots, \nu^{-\frac{1}{2}}\delta(\rho,2); \delta(\nu\rho; T_1(\rho;\sigma)))$$

In this case, $\pi = \pi_1 \oplus \pi_2$.

We note that the case $k-j-1=j=\ell-1$ is a point of irreducibility.

Proof. For $O(2n,F)$ (and $SO(2n+1,F)$, $Sp(2n,F)$), the proof that the reducibility points are as stated is essentially the same argument used in Theorem 4.1, [Jan3]. We will not repeat the arguments here, just restrict ourselves to a comment on the most difficult case: the irreducibility of $\zeta(\rho,2\ell-1) \rtimes \zeta_1(\rho,\ell;\sigma)$. The analogous case in [Jan3] is covered by Lemma 4.3, [Jan3]. A more efficient argument is given in section 6, [Jan5]. For $\zeta(\rho,2\ell-1) \rtimes \zeta_1(\rho,\ell;\sigma)$, the [Jan5] argument only works for $\ell>2$. Thus, the most efficient way to deal with this case seems to be to use [Gol1], [Gol2] for $\ell=1$, a Jacquet module argument similar (but simpler) than that of Lemma 4.3, [Jan3] for $\ell=2$, then the [Jan5] argument for $\ell>2$.

The proof that the π has the irreducible subquotients indicated is similar to the proof of Theorem 6.1, [Jan3]. It is an inductive argument, with the induction on

$k+\ell$ (the parabolic rank of the supercuspidal support). As it is a rather lengthy argument, and there are no new ideas involved, we do not go through the details. Instead, we just restrict ourselves to a few remarks.

As in [Jan3], the induction focuses on $s_{(m)}\pi$. For cases (2) and (3) (noting that case (2) is essentially $j=-1$), we observe that $\pi = \nu^{\frac{-k+1}{2}+j}\zeta(\rho,k) \rtimes \zeta_1(\rho,\ell;\sigma)$ has

$$\begin{aligned} s_{(m)}\pi \quad &= \nu^{-k+j+1}\rho \otimes \nu^{\frac{-k}{2}+j+1}\zeta(\rho,k-1) \rtimes \zeta_1(\rho,\ell;\sigma) \\ &+\nu^{-j}\rho \otimes \nu^{\frac{-k}{2}+j}\zeta(\rho,k-1) \rtimes \zeta_1(\rho,\ell;\sigma) \\ &+\nu^{-\ell+1}\rho \otimes \nu^{\frac{-k+1}{2}+j}\zeta(\rho,k) \rtimes \zeta_1(\rho,\ell-1;\sigma). \end{aligned}$$

We let

$$\begin{aligned} \tau' &= \nu^{-k+j+1}\rho \otimes \nu^{\frac{-k}{2}+j+1}\zeta(\rho,k-1) \rtimes \zeta_1(\rho,\ell;\sigma), \\ \tau'' &= \nu^{-j}\rho \otimes \nu^{\frac{-k}{2}+j}\zeta(\rho,k-1) \rtimes \zeta_1(\rho,\ell;\sigma), \\ \tau''' &= \nu^{-\ell+1}\rho \otimes \nu^{\frac{-k+1}{2}+j}\zeta(\rho,k) \rtimes \zeta_1(\rho,\ell-1;\sigma). \end{aligned}$$

For τ', we have $k'=k-1$, $j'=j$, $\ell'=\ell$ (in the obvious notation) so that $k'-j'-1 = k-j-2$, $j'=j$, $\ell'-1=\ell-1$. Similarly, for τ'' we have $k''-j''-1 = k-j-1$, $j''=j-1$, $\ell''-1=\ell-1$ and for τ''', $k'''-j'''-1 = k-j-1$, $j'''=j$, $\ell'''-1=\ell-2$. Further, by inductive hypothesis, we know that τ', τ'', τ''' decompose according to the theorem.

The proof of the theorem is broken into subcases based on how τ', τ'', τ''' decompose (with respect to the theorem). The particular case of the theorem governing the decomposition of τ' is given in the second column in the table below, and is easily determined from $k'-j'-1$, j', $\ell'-1$. One note: if $j=\frac{k+1}{2}$, in order to avoid having $\alpha'>0$, we replace $\tau' = \nu^{\frac{-k+1}{2}}\rho \otimes \nu^{\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma)$ with $\nu^{\frac{-k+1}{2}}\rho \otimes \nu^{-\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma) = \tau''$ (so then $k'-j'-1 = k''-j''-1$, $j'=j''$, $\ell'-1=\ell''-1$). The third and fourth columns have the corresponding information for τ'' and τ''', respectively. The final column indicates which components of τ', τ'', τ''' are contained in $s_{(m)}\pi_i$ for each component π_i of π. Note that this is part of the induction; we assume the table gives the Jacquet modules for lower values of $k+\ell$ and verifies it for the $k+\ell$ under consideration.

We note that the notation in the tables is the obvious notation; e.g., if τ' decomposes according to case 3a, then τ'_2 is the second component in part 3a of the statement of the theorem.

The proof that the composition series have the indicated structure is similar to the proof of Theorem 7.1, [Jan3]. Again, since there are no new ideas involved, we omit the details.

Table 1. Let $\pi = \nu^{\alpha}\zeta(\rho,k) \rtimes \zeta_1(\rho,\ell;\sigma)$ be as in the statement of Theorem 3.4. The components of π are denoted by $\pi_1,\pi_2,\ldots$; the corresponding Jacquet modules are $s_{(m)}\pi_1, s_{(m)}\pi_2,\ldots$. An explicit description of $\pi_1,\pi_2,\ldots$ is given in Theorem 3.4. The representations τ', τ'', τ''' are the induced representations arising in $s_{(m)}\pi$ ($s_{(m)}\pi = \tau'+\tau''+\tau'''$), possibly reducible. They are described in the proof of Theorem 3.4 and decompose as indicated in the table; the final column indicates which components of τ', τ'', τ''' are contained in $s_{(m)}\pi_i$ for each component π_i of π.

Case (for π)	for τ'	for τ''	for τ'''	components
1. (α) $k=2$, $\alpha=-(\ell+\frac{1}{2})$	3.1	irr	irr	$s_{(m)}\pi_1=\tau'_1$ $s_{(m)}\pi_2=\tau'_2+\tau''+\tau'''$
(β) $k=2$, $\alpha=-(\ell-\frac{1}{2})$	irr	3.1	1	$s_{(m)}\pi_1=\tau'+\tau''_1+\tau'''_1$ $s_{(m)}\pi_2=\tau''_2+\tau'''_2$
(γ) $k>2$, $\alpha=-(\ell+\frac{k-1}{2})$	1	irr	irr	$s_{(m)}\pi_1=\tau'_1$ $s_{(m)}\pi_2=\tau'_2+\tau''+\tau'''$
(δ) $k>2$, $\alpha=-(\ell+\frac{-k+1}{2})$	irr	1	1	$s_{(m)}\pi_1=\tau'+\tau''_1+\tau'''_1$ $s_{(m)}\pi_2=\tau''_2+\tau'''_2$
(ϵ) $k>2$, $-(\ell+\frac{k-1}{2})<\alpha$, $\alpha<-(\ell+\frac{-k+1}{2})$	1	1	1	$s_{(m)}\pi_1=\tau'_1+\tau''_1+\tau'''_1$ $s_{(m)}\pi_2=\tau'_2+\tau''_2+\tau'''_2$
2a.				
(α) $\ell=1,k=2$	3.1	irr	irr	$s_{(m)}\pi_1=\tau'_1$ $s_{(m)}\pi_2=\tau'_2+\tau''$ $s_{(m)}\pi_3=\tau'_3+\tau'''$
(β) $\ell=1,k>2$	2a	irr	irr	$s_{(m)}\pi_1=\tau'_1$ $s_{(m)}\pi_2=\tau'_2+\tau''$ $s_{(m)}\pi_3=\tau'_3+\tau'''$
2b.				
(α) $\ell=2,k=2$	3.1	3.1	2a	$s_{(m)}\pi_1=\tau'_1+\tau'''_1$ $s_{(m)}\pi_2=\tau'_2+\tau''_1$ $s_{(m)}\pi_3=\tau''_2+\tau'''_2$ $s_{(m)}\pi_4=\tau'''_3$
(β) $\ell=2,k>2$	2b	1	2a	$s_{(m)}\pi_1=\tau'_1+\tau'''_1$ $s_{(m)}\pi_2=\tau'_2+\tau''_1$ $s_{(m)}\pi_3=\tau'_3+\tau''_2+\tau'''_2$ $s_{(m)}\pi_4=\tau'_4+\tau'''_3$
(γ) $\ell>2,k=\ell$	2c	1	2b	$s_{(m)}\pi_1=\tau'_1+\tau'''_1$ $s_{(m)}\pi_2=\tau'_2+\tau''_1+\tau'''_2$ $s_{(m)}\pi_3=\tau''_2+\tau'''_3$ $s_{(m)}\pi_4=\tau'''_4$
(δ) $\ell>2,k>\ell$	2b	1	2b	$s_{(m)}\pi_1=\tau'_1+\tau'''_1$ $s_{(m)}\pi_2=\tau'_2+\tau''_1+\tau'''_2$ $s_{(m)}\pi_3=\tau'_3+\tau''_2+\tau'''_3$ $s_{(m)}\pi_4=\tau'_4+\tau'''_4$
2c.				
(α) $k=2,\ell=3$	3.1	irr	2b	$s_{(m)}\pi_1=\tau'_1+\tau'''_1+\tau'''_4$ $s_{(m)}\pi_2=\tau'_2+\tau''+\tau'''_2+\tau'''_3$
(β) $k>2,\ell=k+1$	2d	irr	2b	$s_{(m)}\pi_1=\tau'_1+\tau'''_1+\tau'''_4$ $s_{(m)}\pi_2=\tau'_2+\tau''+\tau'''_2\tau'''_3$

2d.				
(α) $k=2, \ell=4$	3.1	irr	2c	$s_{(m)}\pi_1=\tau_1'+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau''+\tau_2'''$
(β) $k=2, \ell>4$	3.1	irr	2d	$s_{(m)}\pi_1=\tau_1'+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau''+\tau_2'''$
(γ) $k>2, \ell=k+2$	2d	irr	2c	$s_{(m)}\pi_1=\tau_1'+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau''+\tau_2'''$
(δ) $k>2, \ell>k+2$	2d	irr	2d	$s_{(m)}\pi_1=\tau_1'+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau''+\tau_2'''$
3a.				
(α) $\ell=1, j=1, k=4$	3b	3c	3.2	$s_{(m)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau_2'''$ $s_{(m)}\pi_3=\tau_2''+\tau_3'''$
(β) $\ell=1, j=1, k>4$	3a	3c	3.2	$s_{(m)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau_2'''$ $s_{(m)}\pi_3=\tau_3'+\tau_2''+\tau_3'''$
(γ) $\ell=1, j>1, k=2j+2$	3b	3a	3.2	$s_{(m)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau_2''+\tau_2'''$ $s_{(m)}\pi_3=\tau_3''+\tau_3'''$
(δ) $\ell=1, j>1, k>2j+2$	3a	3a	3.2	$s_{(m)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau_2''+\tau_2'''$ $s_{(m)}\pi_3=\tau_3'+\tau_3''+\tau_3'''$
(ϵ) $\ell>1, j=\ell, k=2j+2$	3b	3c	3a	$s_{(m)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau_2'''$ $s_{(m)}\pi_3=\tau_2''+\tau_3'''$
(ζ) $\ell>1, j=\ell, k>2j+2$	3a	3c	3a	$s_{(m)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau_2'''$ $s_{(m)}\pi_3=\tau_3'+\tau_2''+\tau_3'''$
(η) $\ell>1, j>\ell, k=2j+2$	3b	3a	3a	$s_{(m)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau_2''+\tau_2'''$ $s_{(m)}\pi_3=\tau_3''+\tau_3'''$
(θ) $\ell>1, j>\ell, k>2j+2$	3a	3a	3a	$s_{(m)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau_2''+\tau_2'''$ $s_{(m)}\pi_3=\tau_3'+\tau_3''+\tau_3'''$
3b.				(n.b. $\tau'=\tau''$)
(α) $\ell=1, j=1, k=3$	3c	3c	3.2	$s_{(m)}\pi_1=\tau_1'+\tau_2'+\tau_1''+\tau_1'''$ $s_{(m)}\pi_2=\tau_2''+\tau_2'''$
(β) $\ell=1, j>1, k=2j+1$	3a	3a	3.2	$s_{(m)}\pi_1=\tau_1'+\tau_3'+\tau_1''+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau_2''+\tau_3''+\tau_2'''$
(γ) $\ell>1, j=\ell, k=2j+1$	3c	3c	3b	$s_{(m)}\pi_1=\tau_1'+\tau_2'+\tau_1''+\tau_1'''$ $s_{(m)}\pi_2=\tau_2''+\tau_2'''$
(δ) $\ell>1, j>\ell, k=2j+1$	3a	3a	3b	$s_{(m)}\pi_1=\tau_1'+\tau_3'+\tau_1''+\tau_1'''$ $s_{(m)}\pi_2=\tau_2'+\tau_2''+\tau_3''+\tau_2'''$

3c.				
(α) $j=0, \ell=1, k=2$	irr	3.1	3.2	$s_{(m)}\pi_1 = \tau' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_3'' + \tau_2''' + \tau_3'''$
(β) $j=0, \ell=1, k>2$	3c	2a	3.2	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_3''$ $+ \tau_2''' + \tau_3'''$
(γ) $j>0, \ell=j+1, k=2j+2$	irr	3d	3a	$s_{(m)}\pi_1 = \tau' + \tau_1'' + \tau_2'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_3'' + \tau_4'' + \tau_2''' + \tau_3'''$
(δ) $j>0, \ell=j+1, k>2j+2$	3c	3d	3a	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_2'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_3'' + \tau_4''$ $+ \tau_2''' + \tau_3'''$
3d.				
(α) $j=0, \ell=2, k=3$	3e	2b	3c	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2''$ $s_{(m)}\pi_3 = \tau_3''$ $s_{(m)}\pi_4 = \tau_4'' + \tau_2'''$
(β) $j=0, \ell=2, k>3$	3d	2b	3c	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2''$ $s_{(m)}\pi_3 = \tau_3' + \tau_3''$ $s_{(m)}\pi_4 = \tau_4' + \tau_4'' + \tau_2'''$
(γ) $j=0, \ell>2, k=\ell+1$	3e	2b	3d	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_3'' + \tau_3'''$ $s_{(m)}\pi_4 = \tau_4'' + \tau_4'''$
(δ) $j=0, \ell>2, k>\ell+1$	3d	2b	3d	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_3' + \tau_3'' + \tau_3'''$ $s_{(m)}\pi_4 = \tau_4' + \tau_4'' + \tau_4'''$
(ϵ) $j>0, \ell=j+2, k=2j+3$	3e	3d	3c	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2''$ $s_{(m)}\pi_3 = \tau_3''$ $s_{(m)}\pi_4 = \tau_4'' + \tau_2'''$
(ζ) $j>0, \ell=j+2, k>2j+3$	3d	3d	3c	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2''$ $s_{(m)}\pi_3 = \tau_3' + \tau_3''$ $s_{(m)}\pi_4 = \tau_4' + \tau_4'' + \tau_2'''$
(η) $j>0, \ell>j+2,$ $k=\ell+j+1$	3e	3d	3d	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_3'' + \tau_3'''$ $s_{(m)}\pi_4 = \tau_4'' + \tau_4'''$
(θ) $j>0, \ell>j+2,$ $k>\ell+j+1$	3d	3d	3d	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_3' + \tau_3'' + \tau_3'''$ $s_{(m)}\pi_4 = \tau_4' + \tau_4'' + \tau_4'''$

3e.				
(α) $j=0, \ell=2, k=2$	irr	3.1	3c	$s_{(m)}\pi_1 = \tau' + \tau_1'' + \tau_1''' + \tau_2'''$ $s_{(m)}\pi_2 = \tau_2''$
(β) $j=0, \ell>2, k=\ell$	3f(i)	2c	3d	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1''' + \tau_4'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2''' + \tau_3'''$
(γ) $j>0, \ell=j+2,$ $k=\ell+j$	3g	3e	3c	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1''' + \tau_2'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2''$
(δ) $j>0, \ell>j+2,$ $k=\ell+j$	3f(ii)	3e	3d	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1''' + \tau_4'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_3' + \tau_2''$ $+\tau_2''' + \tau_3'''$
3f(i).				
(α) $j=0, k=2, \ell=3$	irr	3.1	3e	$s_{(m)}\pi_1 = \tau' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_2'''$
(β) $j=0, k=2, \ell>3$	irr	3.1	3f(i)	$s_{(m)}\pi_1 = \tau' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_2'''$
(γ) $j=0, k>2, \ell=k+1$	3f(i)	2d	3e	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$
(δ) $j=0, k>2, \ell>k+1$	3f(i)	2d	3f(i)	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$
3f(ii).				
(α) $j=1, k=4, \ell=4$	3g	3f(i)	3e	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_2'$
(β) $j=1, k=4, \ell>4$	3g	3f(i)	3f(ii)	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_2' + \tau_3'''$
(γ) $j=1, k>4, \ell=k$	3f(ii)	3f(i)	3e	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_3'$
(δ) $j=1, k>4, \ell>k$	3f(ii)	3f(i)	3f(ii)	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_3' + \tau_3'''$
(ϵ) $j>1, k=2j+2,$ $\ell=j+3$	3g	3f(ii)	3e	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_2' + \tau_3''$
(ζ) $j>1, k=2j+2,$ $\ell>j+3$	3g	3f(ii)	3f(ii)	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_2' + \tau_3'' + \tau_3'''$
(η) $j>1, k>2j+2,$ $\ell=k-j+1$	3f(ii)	3f(ii)	3e	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_3' + \tau_3''$
(θ) $j>1, k>2j+2,$ $\ell>k-j+1$	3f(ii)	3f(ii)	3(ii)	$s_{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_3' + \tau_3'' + \tau_3'''$

3g.				(n.b. $\tau' = \tau''$)
(α) $j=1, k=3, \ell=3$	3f(i)	3f(i)	irr	$s_{(m)}\pi_1 = \tau_1' + \tau_2' + \tau_1'' + \tau'''$ $s_{(m)}\pi_2 = \tau_2''$
(β) $j=1, k=3, \ell>3$	3f(i)	3f(i)	3g	$s_{(m)}\pi_1 = \tau_1' + \tau_2' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_2'''$
(γ) $j>1, k=2j+1,$ $\ell = j+2$	3f(ii)	3f(ii)	irr	$s_{(m)}\pi_1 = \tau_1' + \tau_2' + \tau_1'' + \tau'''$ $s_{(m)}\pi_2 = \tau_3' + \tau_2'' + \tau_3''$
(δ) $j>1, k=2j+1,$ $\ell > j+2$	3f(ii)	3f(ii)	3g	$s_{(m)}\pi_1 = \tau_1' + \tau_2' + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_3' + \tau_2'' + \tau_3'' + \tau_2'''$

□

Before proceeding to the next result, we pause to make a couple of observations about the preceding theorem.

For case 3, we write $\alpha = \frac{-k+1}{2} + j$, $0 \leq j \leq \frac{k-1}{2}$. As noted in the proof, we may also write case 2 this way, using $j=-1$. Now, $s_{(m)}\pi$ is the sum of three terms (see proof) which are tensor products whose first factors are $\nu^{-(k-j-1)}\rho, \nu^{-j}\rho, \nu^{-(\ell-1)}\rho$, resp. The relations among the exponents $k-j-1, j, \ell-1$ govern the decomposition of 2 and 3 into cases. The reader may observe the similarity between neighboring cases; e.g., one observes that taking $\ell - 1 = j$ in case 3(d), suitably interpreted, gives the irreducible subquotients for 3(c). To make this comparison, Zelevinsky segments (resp., sequences of generalized Steinbergs) of length -1 should be treated as missing; segments (resp., sequences of generalized Steinbergs) of length < -1 should have the entire representation treated as missing.

We also make an observation regarding the generalized Steinbergs which appear in the Langlands data. First, we note that any term appearing in the minimal Jacquet module $s_{min}\pi$ (i.e., the Jacquet module with respect to the smallest standard parabolic subgroup having nonzero Jacquet module) has the form

$$\nu^{x_1}\rho \otimes \nu^{x_2}\rho \otimes \cdots \otimes \nu^{x_{k+\ell}}\rho \otimes \sigma$$

with $\nu^{x_1}\rho \otimes \nu^{x_2}\rho \otimes \cdots \otimes \nu^{x_{k+\ell}}\rho$ a shuffle (i.e., a permutation preserving the relative orders; cf. section 4 [K-R]) of $\nu^{-k+j+1}\rho \otimes \nu^{-k+j+2}\rho \otimes \cdots \otimes \nu^{x}\rho$, $\nu^{-j}\rho \otimes \nu^{-j+1}\rho \otimes \cdots \otimes \nu^{-x-1}\rho$, and $\nu^{-\ell+1}\rho \otimes \nu^{-\ell+2}\rho \otimes \cdots \otimes \rho$. Therefore, one can have at most three consecutive x_i, x_{i+1}, x_{i+2} which are decreasing. This is why we do not get $\delta(\rho, n)$ for $n > 3$. Similar but subtler considerations may be used to constrain the possible tempered representations of orthogonal groups which appear.

We now turn to what might be considered a generalized version of ramified degenerate principal series. Here, we need a bit of additional notation. Suppose $\rho \not\cong \rho_0$ are representations of $GL(m,F)$ and $GL(m_0,F)$, with both (ρ,σ) and (ρ_0,σ) satisfying (C0). Again, let $\rho \rtimes \sigma = T_1(\rho;\sigma) + T_2(\rho;\sigma)$ and $\rho_0 \rtimes \sigma = T_1(\rho_0;\sigma) + T_2(\rho_0;\sigma)$. By [Gol1], [Gol2], $\rho \rtimes T_i(\rho_0;\sigma)$ and $\rho_0 \rtimes T_j(\rho;\sigma)$ have a common component. Denote this common component by $T_{i,j}(\rho_0,\rho;\sigma)$.

Theorem 3.5. *Suppose that ρ, ρ_0 are irreducible unitary supercuspidal representations of $GL(m,F), GL(m_0,F)$ and σ an irreducible supercuspidal representation of $G(r,F)$ such that both (ρ,σ) and (ρ_0,σ) satisfy (C0). Let $\pi = \nu^{\alpha}\zeta(\rho_0,k) \rtimes \zeta_1(\rho,\ell;\sigma)$ with $\alpha \in \mathbb{R}$, $k \geq 1$. Then π is reducible if and only if $\alpha \in \{\frac{-k+1}{2}, \frac{-k+3}{2}, \ldots, \frac{k-1}{2}\}$. Suppose π is reducible. By contragredience, we may assume that $\alpha \leq 0$. Write $\alpha = \frac{-k+1}{2} + j$ with $0 \leq j \leq \frac{k-1}{2}$.*

(1) $j = \frac{k-1}{2}$
with $\pi = \pi_1 + \pi_2$

$$\pi_i = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-1}\rho_0], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-1}\rho_0]; T_{i,1}(\rho_0, \rho; \sigma))$$

for $i = 1, 2$. *In this case,* $\pi = \pi_1 \oplus \pi_2$.

(2) $0 \leq j < \frac{k-1}{2}$
$\pi = \pi_1 + \pi_2 + \pi_3$ *with*

$$\pi_i = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{-k+j+1}\rho_0, \nu^{-1}\rho_0], [\nu^{-j}\rho_0, \nu^{-1}\rho_0]; T_{i,1}(\rho_0, \rho; \sigma))$$

for $i = 1, 2$ *and*

$$\pi_3 = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{-k+j+1}\rho_0, \nu^{-j-2}\rho_0], \nu^{-j-\frac{1}{2}}\delta(\rho_0, 2), \nu^{-j+\frac{1}{2}}\delta(\rho_0, 2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho_0, 2); T_1(\rho; \sigma)).$$

In this case, π_3 *is the unique irreducible quotient and* $\pi_1 \oplus \pi_2$ *is a subrepresentation.*

Proof. For $O(2n, F)$ (and $SO(2n+1, F)$, $Sp(2n, F)$), the arguments from section 5, [Jan3] may be used to identify the irreducible subquotients and their Jacquet modules. Alternatively, and more directly, they may be obtained from the results of [Jan6]. The arguments from section 7, [Jan3] may be used to determine the composition series. □

Remark 3.6. Let ρ and σ be as in the preceding theorem. Suppose ρ_0 is an irreducible unitary supercuspidal representation of $GL(m_0, F)$ with $\rho_0 \not\cong \tilde{\rho}_0$. Then, $\nu^\alpha \zeta(\rho_0, k) \rtimes \zeta_1(\rho, \ell; \sigma)$ is irreducible for all $\alpha \in \mathbb{R}$. (This also follows from [Jan6] or an argument like that in [Jan3].)

4. Restrictions of representations

Our analysis of generalized degenerate principal series for $SO(2n, F)$ is based on our results for $O(2n, F)$ and the connection between induced representations for $SO(2n, F)$ and those for $O(2n, F)$. To establish the connection between generalized degenerate principal series for $G^0 = SO(2n, F)$ and $G = O(2n, F)$, we study the restriction from G to G^0 in general (Lemmas 4.1 and 4.2), for Langlands data (Lemma 4.6) and for representations $T_i(\rho; \sigma)$ and $\zeta_i(\rho, \ell; \sigma)$. Proposition 4.3 and Corollary 4.4 discuss cuspidal reducibility. In Lemma 4.5, we describe the connection between composition series for representations of G and G^0.

Recall that for $n \geq 1$,

$$O(2n, F) = SO(2n, F) \rtimes \{1, s\},$$

where s is defined in section 2. We denote by $\hat{s}$ the nontrivial character of $O(2n, F)$ defined by

$$\hat{s}(g) = 1,$$
$$\hat{s}(gs) = -1,$$

for every $g \in SO(2n, F)$.

Lemma 4.1. *Let* $G = O(2n, F)$, $G^0 = SO(2n, F)$, *with* $n > 0$.

(1) *For any admissible representation π_0 of G^0 and any admissible representation π of G (π_0, π not necessarily irreducible), we have*

$$r_{G^0,G} \circ i_{G,G^0}(\pi_0) \cong \pi_0 \oplus s\pi_0, \qquad i_{G,G^0}(s\pi_0) \cong i_{G,G^0}(\pi_0),$$
$$i_{G,G^0} \circ r_{G^0,G}(\pi) \cong \pi \oplus \hat{s}\pi, \qquad r_{G^0,G}(\hat{s}\pi) \cong r_{G^0,G}(\pi).$$

(2) *Let σ be an irreducible admissible representation of G. Suppose that σ_0 is an irreducible subquotient of $r_{G^0,G}(\sigma)$. Then,*

$$\sigma_0 \cong s\sigma_0 \text{ if and only if } \sigma \ncong \hat{s}\sigma.$$

(a) *If $\sigma_0 \cong s\sigma_0$, then*

$$i_{G,G^0}(\sigma_0) \cong \sigma \oplus \hat{s}\sigma,$$
$$r_{G^0,G}(\sigma) \cong \sigma_0.$$

(b) *If $\sigma_0 \ncong s\sigma_0$, then*

$$i_{G,G^0}(\sigma_0) \cong \sigma,$$
$$r_{G^0,G}(\sigma) \cong \sigma_0 \oplus s\sigma_0.$$

Proof. 1. The first statement follows from [B-Z], Theorem 5.2, noting that since both G^0 and sG^0 are open in G, either can start the filtration, hence both π_0 and $s\pi_0$ appear as subrepresentations.

An isomorphism $i_{G,G^0} \circ r_{G^0,G}(\pi) \cong \pi \oplus \hat{s}\pi$ can be constructed in the following way: Let V denote the space of π. For $v \in V$, define $\varphi_v : G \to V$ and $\psi_v : G \to V$ by

$$\varphi_v(x) = \pi(x)v, \qquad \psi_v(x) = \hat{s}\pi(x)v.$$

Then $\varphi : V \to i_{G,G^0}(V)$ given by $\varphi(v) = \varphi_v$ and $\psi : V \to i_{G,G^0}(V)$ defined by $\psi(v) = \psi_v$ are intertwining operators and $\varphi \oplus \psi$ is an isomorphism between $\pi \oplus \hat{s}\pi$ and $i_{G,G^0} \circ r_{G^0,G}(\pi)$.

The proof that $i_{G,G^0}(s\pi_0) \cong i_{G,G^0}(\pi_0)$ is direct, using the intertwining operator φ between $i_{G,G^0}(\pi_0)$ and $i_{G,G^0}(s\pi_0)$ given by

$$\varphi(f)(g) = f(sg),$$

where $f \in i_{G,G^0}(\pi_0)$, $g \in G$.

The last statement follows from [B-Z], Proposition 1.9.

2. This follows from the results in section 2, [G-K] (cf. Lemma 2.1, [B-J1]). $\square$

Lemma 4.2. *Let σ be an admissible representation of $O(2m,F)$, $m > 0$, and σ_0 an admissible representation of $SO(2m,F)$. Let ρ be an admissible representation of $GL(n,F)$. Set $G = O(2(m+n),F)$, $G^0 = SO(2(m+n),F)$.*

(1)

$$\hat{s}(\rho \rtimes \sigma) \cong \rho \rtimes \hat{s}\sigma, \qquad s(\rho \rtimes \sigma_0) \cong \rho \rtimes s\sigma_0.$$

(2) *Suppose that σ is irreducible and σ_0 is an irreducible subquotient of $r_{SO(2m,F),O(2m,F)}(\sigma)$.*

(a) *If $\sigma_0 \cong s\sigma_0$, then*

$$i_{G,G^0}(\rho \rtimes \sigma_0) = \rho \rtimes \sigma + \hat{s}(\rho \rtimes \sigma),$$
$$r_{G^0,G}(\rho \rtimes \sigma) = \rho \rtimes \sigma_0.$$

(b) *If* $\sigma_0 \not\cong s\sigma_0$, *then*

$$i_{G,G^0}(\rho \rtimes \sigma_0) = \rho \rtimes \sigma,$$
$$r_{G^0,G}(\rho \rtimes \sigma) = \rho \rtimes \sigma_0 + \rho \rtimes s\sigma_0.$$

Proof. 1. The first statement follows from [B-Z], Proposition 1.9, the second from [Ban2], Corollary 4.1.

2. Let M^0 (respectively, M) be the standard Levi subgroup of G^0 (respectively, G) isomorphic to $GL(n,F) \times SO(2m,F)$ (respectively, $GL(n,F) \times O(2m,F)$).

(a) Suppose $\sigma_0 \cong s\sigma_0$. Then $\rho \rtimes \sigma_0 \cong s(\rho \rtimes \sigma_0)$. According to Lemma 4.1, $i_{O(2m,F),SO(2m,F)}(\sigma_0) = \sigma + \hat{s}\sigma$. We have

$$\begin{aligned}\rho \rtimes \sigma + \hat{s}(\rho \rtimes \sigma) &= i_{G,M}(\rho \otimes \sigma) + i_{G,M}(\rho \otimes \hat{s}\sigma)\\ &= i_{G,M}(\rho \otimes (\sigma + \hat{s}\sigma)) = i_{G,M^0}(\rho \otimes \sigma_0) = i_{G,G^0}(\rho \rtimes \sigma_0).\end{aligned}$$

By Lemma 4.2,

$$r_{G^0,G} \circ i_{G,G^0}(\rho \rtimes \sigma_0) = \rho \rtimes \sigma_0 + s(\rho \rtimes \sigma_0) = 2\rho \rtimes \sigma_0.$$

On the other hand,

$$r_{G^0,G} \circ i_{G,G^0}(\rho \rtimes \sigma_0) = r_{G^0,G}(\rho \rtimes \sigma) + r_{G^0,G}(\hat{s}(\rho \rtimes \sigma)) = 2r_{G^0,G}(\rho \rtimes \sigma).$$

It follows that $r_{G^0,G}(\rho \rtimes \sigma) = \rho \rtimes \sigma_0$.

(b) Suppose that $\sigma_0 \not\cong s\sigma_0$. According to Lemma 4.1, $i_{O(2m,F),SO(2m,F)}(\sigma_0) = \sigma$. It follows that

$$\rho \rtimes \sigma = i_{G,M}(\rho \otimes \sigma) = i_{G,M^0}(\rho \otimes \sigma_0) = i_{G,G^0}(\rho \rtimes \sigma_0).$$

Further,

$$r_{G^0,G} \circ i_{G,G^0}(\rho \rtimes \sigma_0) = \rho \rtimes \sigma_0 + \rho \rtimes s\sigma_0$$

and

$$r_{G^0,G} \circ i_{G,G^0}(\rho \rtimes \sigma_0) = r_{G^0,G}(\rho \rtimes \sigma).$$

It follows that $r_{G^0,G}(\rho \rtimes \sigma) = \rho \rtimes \sigma_0 + \rho \rtimes s\sigma_0$. □

The following proposition relates the reducibility of $\nu^x\rho \rtimes \sigma$ and that of $\nu^x\rho \rtimes \sigma_0$ (for $x \in \mathbb{R}$). This proposition tells us the conditions under which (ρ, σ) satisfy (C0) implies (ρ, σ_0) satisfy (C0) (noting the relevence of the latter to generalized degenerate principal series for $SO(2n,F)$ discussed in section 3). In addition, as a corollary, we deduce that cuspidal reducibility for $O(2n,F)$ has a characterization like that for $SO(2n+1,F)$, $Sp(2n,F)$, and $SO(2n,F)$.

Proposition 4.3. *Suppose ρ is an irreducible unitary supercuspidal representation of $GL(m,F)$ and σ an irreducible supercuspidal representation of $O(2r,F)$. Suppose σ_0 is an irreducible subquotient of $r_{SO(2r,F),O(2r,F)}(\sigma)$.*

(1) *$r > 0$ and $s \cdot \sigma_0 \cong \sigma_0$.*

For all $x \in \mathbb{R}$, we have $\nu^x\rho \rtimes \sigma$ is reducible if and only if $\nu^x\rho \rtimes \sigma_0$ is reducible.

(2) *$r = 0$ or $\sigma_0 \not\cong s \cdot \sigma_0$.*

Here, there are two possibilities:

(a) *m odd with $\rho \cong \tilde{\rho}$.*

In this case, $\nu^x\rho \rtimes \sigma_0$ is irreducible for all $x \in \mathbb{R}$. However, $\nu^x\rho \rtimes \sigma$ is irreducible for all $x \in \mathbb{R} \setminus \{0\}$ and reducible for $x = 0$.

(b) *m even or $\rho \ncong \tilde{\rho}$.*
For all $x \in \mathbb{R}$, we have $\nu^x\rho \rtimes \sigma$ is reducible if and only if $\nu^x\rho \rtimes \sigma_0$ is reducible.

Proof. Let M be the standard Levi subgroup of G isomorphic to $GL(m, F) \times O(2r, F)$. For (1), we show that $\nu^x\rho \rtimes \sigma$ is irreducible if and only if $\nu^x\rho \rtimes \sigma_0$ is irreducible. For (2), we show two things: (i) $\nu^x\rho \rtimes \sigma$ irreducible implies $\nu^x\rho \rtimes \sigma_0$ irreducible, and (ii) $\nu^x\rho \rtimes \sigma_0$ irreducible implies $\nu^x\rho \rtimes \sigma$ irreducible unless m is odd, $x = 0$, and $\rho \cong \tilde{\rho}$, in which case $\nu^x\rho \rtimes \sigma$ reduces.

We start with (1), so we may assume $r > 0$ and $\sigma_0 \cong s\sigma_0$. By Lemma 4.2,

$$r_{G^0,G}(\nu^x\rho \rtimes \sigma) = \nu^x\rho \rtimes \sigma_0.$$

Therefore, if $\nu^x\rho \rtimes \sigma_0$ is irreducible, then $\nu^x\rho \rtimes \sigma$ is irreducible. On the other hand, suppose that $\nu^x\rho \rtimes \sigma$ is irreducible. We have

$$r_{M,G}(\nu^x\rho \rtimes \sigma) = \nu^x\rho \otimes \sigma + \nu^{-x}\tilde{\rho} \otimes \sigma.$$

Since $\sigma \ncong \hat{s}\sigma$, Lemma 4.2(1) implies

$$r_{M,G}(\hat{s}(\nu^x\rho \rtimes \sigma)) = \nu^x\rho \otimes \hat{s}\sigma + \nu^{-x}\tilde{\rho} \otimes \hat{s}\sigma.$$

It follows that $\nu^x\rho \rtimes \sigma \ncong \hat{s}(\nu^x\rho \rtimes \sigma)$. By Lemma 4.1, $\nu^x\rho \rtimes \sigma_0 = r_{G^0,G}(\nu^x\rho \rtimes \sigma)$ is irreducible.

We now address (2) for $r > 0$, so we may assume $r > 0$ and $\sigma_0 \ncong s\sigma_0$. By Lemma 4.2,

$$i_{G,G^0}(\nu^x\rho \rtimes \sigma_0) = \nu^x\rho \rtimes \sigma.$$

Therefore, if $\nu^x\rho \rtimes \sigma$ is irreducible, then $\nu^x\rho \rtimes \sigma_0$ is irreducible. On the other hand, suppose that $\nu^x\rho \rtimes \sigma_0$ is irreducible. We have

$$r_{M^0,G^0}(\nu^x\rho \rtimes \sigma_0) = \nu^x\rho \otimes \sigma_0 + \nu^{-x}\tilde{\rho} \otimes s^m\sigma_0.$$

By Lemma 4.2(1),

$$r_{M^0,G^0}(s(\nu^x\rho \rtimes \sigma_0)) = \nu^x\rho \otimes s\sigma_0 + \nu^{-x}\tilde{\rho} \otimes s^{m+1}\sigma_0.$$

Thus, $\nu^x\rho \rtimes \sigma_0 \ncong s(\nu^x\rho \rtimes \sigma_0)$ unless m is odd, $x = 0$, and $\rho \cong \tilde{\rho}$. By Lemma 4.1, $\nu^x\rho \rtimes \sigma = i_{G,G^0}(\nu^x\rho \rtimes \sigma_0)$ is irreducible. If m is odd, $x = 0$, and $\rho \cong \tilde{\rho}$, then Theorem 6.11 of [Gol1] tells us that $\rho \rtimes \sigma_0$ is irreducible and Theorem 3.3 of [Gol2] implies that $\rho \rtimes \sigma$ has two components.

Finally, we address (2) for $r = 0$. Then $\sigma_0 = 1_0$, $\sigma = 1$, both trivial representations of the trivial group.

First, since $M = M^0$, we have

$$i_{G,M}(\nu^x\rho \otimes 1) \cong i_{G,G^0} \circ i_{G^0,M^0}(\nu^x\rho \otimes 1_0).$$

Therefore, if $i_{G,M}(\nu^x\rho \otimes 1)$ is irreducible, then so is $i_{G^0,M^0}(\nu^x\rho \otimes 1_0)$.

On the other hand, suppose $i_{G^0,M^0}(\nu^x\rho \otimes 1_0)$ is irreducible. Suppose $x \neq 0$; without loss of generality, $x < 0$. Then, $i_{G^0,M^0}(\nu^x\rho \otimes 1_0) = L(\nu^x\rho \otimes 1_0)$. Since

$$s \cdot L(\nu^x\rho) = L(s \cdot (\nu^x\rho \otimes 1_0)) \ncong L(\nu^x\rho \otimes 1_0),$$

we see that $i_{G,G^0} \circ i_{G^0,M^0}(\nu^x\rho \otimes 1_0)$ is irreducible, as needed. When $x = 0$, Theorems 3.1 and 3.3 of [Gol2] tell us that $i_{G^0,M^0}(\rho \otimes 1_0)$ and $i_{G,M}(\rho \otimes 1)$ have the same number of components (implying the irreducibility of $i_{G,M}(\rho \otimes 1)$) unless ρ is a self-contragredient representation of $GL_m(F)$ with m odd, in which $i_{G,M}(\rho \otimes 1)$ has twice as many components as $i_{G^0,M^0}(\rho \otimes 1_0)$ (implying the reducibility of $i_{G,M}(\rho \otimes 1)$), as needed. $\square$

It is worth noting that the results may be interpreted as follows: If $\nu^x\rho \rtimes \sigma$ is irreducible, then $\nu^x\rho \rtimes \sigma_0$ is irreducible. On the other hand, if $\nu^x\rho \rtimes \sigma_0$ is irreducible, then $\nu^x\rho \rtimes \sigma$ is irreducible unless $\nu^x\rho \otimes \sigma$ is unitary (so $x = 0$) and the pair $(P, \rho \otimes \sigma)$ is ramified (in the sense of Harish-Chandra, cf. [Sil2]) but $(P^0, \rho \otimes \sigma_0)$ is unramified.

Corollary 4.4. *Assume the conjectures needed for* [Mœ] *or* [Zh]. *With notation as above, we have the following:*

(1) *If $\rho \not\cong \tilde{\rho}$, then $\nu^x\rho \rtimes \sigma$ is irreducible for all $x \in \mathbb{R}$.*
(2) *If $\rho \cong \tilde{\rho}$, then there is a unique $\alpha \geq 0$ with $\alpha \in \frac{1}{2}\mathbb{Z}$ such that $\nu^x\rho \rtimes \sigma$ is reducible and $\nu^x\rho \rtimes \sigma$ is irreducible for all $x \in \mathbb{R} \setminus \{\pm\alpha\}$. (The specific value of α may be determined by the preceding theorem and the results of* [Mœ], [Zh].*)*

In section 3, we claimed that when (ρ, σ) and (ρ, σ_0) both satisfy (C0), the composition series for the generalized degenerate principal series $\nu^\alpha\zeta(\rho, k) \rtimes \zeta_1(\rho, \ell; \sigma)$ and those of $\nu^\alpha\zeta(\rho, k) \rtimes \zeta_1(\rho, \ell; \sigma_0)$ have the same form. Lemmas 4.5 and 4.6 are used (in section 5) to show that this is indeed the case.

Lemma 4.5. *Let σ be an irreducible admissible representation of $O(2m, F)$, $m > 0$, and σ_0 an irreducible subquotient of $r_{SO(2m,F),O(2m,F)}(\sigma)$. Let ρ be an admissible representation of $GL(n, F)$. Set $G = O(2(m + n), F)$ and $G^0 = SO(2(m + n), F)$.*

(1) *Suppose that $\sigma_0 \cong s\sigma_0$ and that $\pi_0 \cong s\pi_0$ for every irreducible subquotient π_0 of $\rho \rtimes \sigma_0$. Then $\rho \rtimes \sigma$ and $\rho \rtimes \sigma_0$ have the same number of irreducible subquotients. Assume, in addition, that for any two irreducible subquotients π and π' of $\rho \rtimes \sigma$, $\hat{s}\pi \not\cong \pi'$. Then*
$$0 \subset \pi_1 \subset \cdots \subset \pi_k = \rho \rtimes \sigma$$
is a composition series for $\rho \rtimes \sigma$ if and only if
$$0 \subset r_{G^0,G}(\pi_1) \subset \cdots \subset r_{G^0,G}(\pi_k) = \rho \rtimes \sigma_0$$
is a composition series for $\rho \rtimes \sigma_0$.
(2) *Suppose that $\sigma \cong \hat{s}\sigma$ and that $\pi \cong \hat{s}\pi$ for every irreducible subquotient π of $\rho \rtimes \sigma$. Then $\rho \rtimes \sigma$ and $\rho \rtimes \sigma_0$ have the same number of irreducible subquotients. Assume, in addition, that for any two irreducible subquotients π_0 and π_0' of $\rho \rtimes \sigma_0$, $s\pi_0 \not\cong \pi_0'$. Then*
$$0 \subset \pi_1^0 \subset \cdots \subset \pi_k^0 = \rho \rtimes \sigma_0$$
is a composition series for $\rho \rtimes \sigma_0$ if and only if
$$0 \subset i_{G,G^0}(\pi_1) \subset \cdots \subset i_{G,G^0}(\pi_k) = \rho \rtimes \sigma$$
is a composition series for $\rho \rtimes \sigma$.

Proof. 1. According to Lemma 4.2, $r_{G^0,G}(\rho \rtimes \sigma) = \rho \rtimes \sigma_0$. Let
$$\rho \rtimes \sigma = \pi_1 + \cdots + \pi_k,$$
where $\pi_1, \dots, \pi_k$ are irreducible. For $i = 1, \dots, k$, let π_i^0 be an irreducible subquotient of $r_{G^0,G}(\pi_i)$. Then π_i^0 is an irreducible component of $r_{G^0,G}(\rho \rtimes \sigma) = \rho \rtimes \sigma_0$. Since $\pi_i^0 \cong s\pi_i^0$, we have (Lemma 4.1)
$$r_{G^0,G}(\pi_i) = \pi_i^0.$$
It follows that
$$\rho \rtimes \sigma_0 = r_{G^0,G}(\rho \rtimes \sigma) = r_{G^0,G}(\pi_1 + \cdots + \pi_k) = \pi_1^0 + \cdots + \pi_k^0.$$

We deal with composition series inductively. In order to do this, we have to work in slightly greater generality. To this end, suppose that π is an admissible representation of G such that $\hat{s}\pi \cong \pi$ and $\pi' \cong \hat{s}\pi'$ for every irreducible subquotient π' of π. Further, assume the following: (1) if π', π'' are irreducible subquotients of π, then $\pi' \ncong \hat{s}\pi''$, and (2) if π_i is an irreducible subquotient of π, then π_i^0 is an irreducible subquotient of $\pi^0 = r_{G^0,G}(\pi)$. We note that these assumptions hold when $\pi = \rho \rtimes \sigma$. Now, let π_1 be an irreducible subquotient of π. We prove that π_1 is a subrepresentation of π if and only if π_1^0 is a subrepresentation of π_0. The statement then follows by the induction on the number of irreducible subquotients.

Suppose that π_1 is a subrepresentation of π. Then we have the exact sequence

$$0 \longrightarrow \pi_1 \longrightarrow \pi \longrightarrow \pi/\pi_1 \longrightarrow 0.$$

The functor $r_{G^0,G}$ is exact ([B-Z], Proposition 1.9), so we have the exact sequence

$$0 \longrightarrow r_{G^0,G}(\pi_1) \longrightarrow r_{G^0,G}(\pi) \longrightarrow r_{G^0,G}(\pi/\pi_1) \longrightarrow 0;$$

i.e.,

$$0 \longrightarrow \pi_1^0 \longrightarrow \pi_0 \longrightarrow \pi_0/\pi_1^0 \longrightarrow 0,$$

so π_1^0 is a subrepresentation of π_0.

Conversely, assume $\pi_1^0 \hookrightarrow \pi_0$. Then the exact sequence

$$0 \longrightarrow \pi_1^0 \longrightarrow \pi_0 \longrightarrow \pi/\pi_1^0 \longrightarrow 0$$

implies

$$0 \longrightarrow i_{G,G^0}(\pi_1^0) \longrightarrow i_{G,G^0}(\pi_0) \longrightarrow i_{G,G^0}(\pi_0/\pi_1^0) \longrightarrow 0,$$

so, by Lemma 4.2, we have

$$0 \longrightarrow \pi_1^0 \oplus \hat{s}\pi_1^0 \longrightarrow \pi \oplus \hat{s}\pi \longrightarrow i_{G,G^0}(\pi_0/\pi_1^0) \longrightarrow 0.$$

Therefore, π_1 is a subrepresentation of $\pi \oplus \hat{s}\pi$. By the assumption, π_1 is not a component of $\hat{s}\pi$. We conclude that π_1 is a subrepresentation of π.

The proof of (2) is similar to that of (1). □

Lemma 4.6. *Let ρ_i, $i = 1, \dots, k$ be an irreducible essentially square-integrable representation of $GL(n_i, F)$ and σ an irreducible tempered representation of $O(2m, F)$, $m \geq 0$. If $m > 0$, let σ_0 be an irreducible subquotient of $r_{SO(2m,F),O(2m,F)}(\sigma)$. If $m = 0$, let $\sigma_0 = 1_0$. Set $n = n_1 + \cdots + n_k + m$, $G = O(2n, F)$, $G^0 = SO(2n, F)$. Suppose that $e(\rho_1) \leq \cdots \leq e(\rho_k) < 0$. Then*

$$\tau = \rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma \quad \textit{and} \quad \tau_0 = \rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma_0$$

are Langlands data for G and G^0 and

$$sL(\tau_0) = L(s\tau_0),$$
$$\hat{s}L(\tau) = L(\hat{s}\tau).$$

Moreover, $L(\tau)$ is a component of $i_{G,G^0}(L(\tau_0))$.

Proof. Let M^0 (respectively, M) denote the standard Levi subgroup corresponding to τ_0 (respectively, τ).

The first equality follows from [B-J1], Proposition 4.5. According to Lemma 4.2,

$$i_{G,M}(\hat{s}\tau) \cong \hat{s}i_{G,M}(\tau).$$

Now, $L(\hat{s}\tau)$ is the unique irreducible subrepresentation of $i_{G,M}(\hat{s}\tau)$ and $\hat{s}L(\tau)$ is the unique irreducible subrepresentation of $\hat{s}i_{G,M}(\tau)$. It follows that $\hat{s}L(\tau) \cong L(\hat{s}\tau)$.

To prove that $L(\tau)$ is a component of $i_{G,G^0}(L(\tau_0))$, suppose that $\sigma_0 \cong s\sigma_0$. Then $\tau_0 \cong s\tau_0$, $\tau \ncong \hat{s}\tau$, $i_{M,M^0}(\tau_0) \cong \tau \oplus \hat{s}\tau$. We have

$$i_{G,G^0} \circ i_{G^0,M^0}(\tau_0) \cong i_{G,M}(\tau \oplus \hat{s}\tau) \cong i_{G,M}(\tau) \oplus i_{G,M}(\hat{s}\tau).$$

This representation has two irreducible subrepresentations, $L(\tau)$ and $L(\hat{s}\tau)$. $L(\tau_0)$ is the unique irreducible subrepresentation of $i_{G^0,M^0}(\tau_0)$. Since $sL(\tau_0) \cong L(s\tau_0) \cong L(\tau_0)$, $i_{G,G^0}(L(\tau_0))$ is the direct sum of two irreducible representations. They are subrepresentations of $i_{G,G^0} \circ i_{G^0,M^0}(\tau_0)$. It follows that

$$i_{G,G^0}(L(\tau_0)) \cong L(\tau) \oplus \hat{s}L(\tau) \cong L(\tau) \oplus L(\hat{s}\tau).$$

In the case $\sigma_0 \ncong s\sigma_0$, the proof is similar. □

Lemma 4.7. *Let ρ be an irreducible supercuspidal unitary representation of $GL(n,F)$ and σ an irreducible supercuspidal representation of $O(2m,F)$, $m \geq 0$. If $m > 0$, let σ_0 be an irreducible subquotient of $r_{SO(2m,F),O(2m,F)}(\sigma)$. If $m = 0$, let $\sigma_0 = 1_0$. Set $G = O(2(m+n),F)$, $G^0 = SO(2(m+n),F)$.*

(1) *Suppose that $\rho \rtimes \sigma_0$ is reducible. Let*

$$\rho \rtimes \sigma_0 \cong T_1(\rho;\sigma_0) \oplus T_2(\rho;\sigma_0)$$

be the decomposition of $\rho \rtimes \sigma_0$ into the direct sum of two inequivalent irreducible subrepresentations. Then there exists a decomposition

$$\rho \rtimes \sigma \cong T_1(\rho;\sigma) \oplus T_2(\rho;\sigma)$$

into the direct sum of two inequivalent irreducible subrepresentations such that, for $i = 1,2$, $T_i(\rho;\sigma_0)$ (resp., $\zeta_i(\rho,\ell;\sigma_0)$) is an irreducible subquotient of $r_{G^0,G}(T_i(\rho;\sigma))$ (resp., $r_{SO(2m+2\ell n,F),O(2m+2\ell n,F)}(\zeta_i(\rho,\ell;\sigma))$).

(a) *If $m > 0$ and $\sigma_0 \cong s\sigma_0$, then*

$$s(T_i(\rho;\sigma_0)) \cong T_i(\rho;\sigma_0),$$
$$s(\zeta_i(\rho,\ell;\sigma_0)) \cong \zeta_i(\rho,\ell;\sigma_0).$$

(b) *If $m = 0$ or $m > 0$, $\sigma_0 \ncong s\sigma_0$, then n is even and*

$$s(T_i(\rho;\sigma_0)) \ncong T_i(\rho;\sigma_0),$$
$$s(\zeta_i(\rho,\ell;\sigma_0)) \ncong \zeta_i(\rho,\ell;\sigma_0).$$

(2) *Suppose that n is odd and $\rho \cong \tilde{\rho}$. If $m > 0$, suppose that $\sigma_0 \ncong s\sigma_0$.*

(a) *$\rho \rtimes \sigma_0$ is an irreducible tempered representation and*

$$\rho \rtimes \sigma_0 \cong s(\rho \rtimes \sigma_0).$$

(b) *$\rho \rtimes \sigma$ is reducible and it is the direct sum of two inequivalent tempered representations*

$$\rho \rtimes \sigma \cong T_1(\rho;\sigma) \oplus T_2(\rho;\sigma),$$
$$r_{G^0,G}(T_1(\rho;\sigma)) = r_{G^0,G}(T_2(\rho;\sigma)) = \rho \rtimes \sigma_0,$$
$$r_{G^0,G}(\zeta_1(\rho,\ell;\sigma)) = r_{G^0,G}(\zeta_2(\rho,\ell;\sigma)) = \zeta(\rho,\ell;\sigma_0).$$

5. Degenerate principal series for $SO(2n, F)$

In this section, we deal with generalized degenerate principal series for $SO(2n, F)$. Let ρ, ρ_0 be irreducible unitary supercuspidal representations of $GL(m, F)$, $GL(m_0, F)$. Let σ_0 (resp., σ) be an irreducible supercuspidal representation of $SO(2r, F)$ (resp. $O(2r, F)$) such that σ_0 is a component of $r_{SO(2r,F),O(2r,F)}(\sigma)$. (We are allowing the possibility that $r = 0$ here.) Further, suppose that (ρ, σ) satisfies (C0) (which implies $\rho \cong \tilde{\rho}$). There are two possibilities (cf. Proposition 4.3): (1) (ρ, σ_0) also satisfies (C0), and (2) $\nu^x\rho \rtimes \sigma_0$ is irreducible for all $x \in \mathbb{R}$ (which can happen only if m is odd and either $r = 0$ or $\sigma_0 \not\cong s\sigma_0$). In the first case, the results for $\nu^\alpha\zeta(\rho_0, k) \rtimes \zeta_1(\rho, \ell; \sigma_0)$ are given in section 3. The proofs in section 3 are for $O(2n, F)$; the proofs for $SO(2n, F)$ are handled in this section. In the second case, the results for $\nu^\alpha\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma_0)$ are different than those in section 3; both the statements and proofs are given in this section. In both cases, the proofs are built from the results on $\nu^\alpha\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$ and our study of $r_{G^0,G}, i_{G,G^0}$ from section 4.

We start by dealing with the first case, where (ρ, σ_0) also satisfies (C0). We prove Theorem 3.4. Set

$$\pi_0 = \nu^\alpha\zeta(\rho, k) \rtimes \zeta_1(\rho, \ell; \sigma_0),$$
$$\pi = \nu^\alpha\zeta(\rho, k) \rtimes \zeta_1(\rho, \ell; \sigma).$$

First, consider the case $r \neq 0$.

The irreducible subquotients in Theorem 3.4 are described as Langlands subrepresentations. Let

$$\{L(\tau) \mid \tau \in T\}$$

be the set of all irreducible subquotients appearing in the statement of Theorem 3.4. For $\tau \in T$, denote by τ_0 the representation obtained by replacing σ by σ_0 (noting that since (ρ, σ_0) also satisfies (C0), we have $T_i(\rho, \sigma_0)$ and $\delta(\nu\rho; T_i(\rho, \sigma_0))$ defined). We have two possibilities: $\sigma_0 \cong s\sigma_0$ or $\sigma_0 \not\cong s\sigma_0$.

1. Suppose that $\sigma_0 \cong s\sigma_0$. Then $\zeta_1(\rho, \ell; \sigma_0) \cong s\zeta_1(\rho, \ell; \sigma_0)$ (Lemma 4.7) and $\pi_0 \cong s\pi_0$ (Lemma 4.2). By inspecting all $\tau \in T$, we conclude that $s\tau_0 \cong \tau_0$ and

$$sL(\tau_0) \cong L(s\tau_0) \cong L(\tau_0).$$

Also, for $\tau, \tau' \in T$ we have

$$L(\tau) \not\cong \hat{s}L(\tau').$$

Therefore, we are in the situation described by Lemma 4.5, (1). According to Lemma 4.6,

$$L(\tau_0) \cong r_{G^0,G}(L(\tau)).$$

We conclude that π and π_0 have the same number of irreducible components and the same composition series structure.

Note that

$$r_{G^0,G}(\pi) = \pi_0$$

and

$$r_{M^0,G^0}(\pi_0) = r_{M^0,G^0} \circ r_{G^0,G}(\pi) = r_{M^0,M} \circ r_{M,G}(\pi).$$

Therefore, Jacquet modules of π_0 can be found by taking restrictions of Jacquet modules of π. Let τ', τ'', τ''' be as in the proof of Theorem 3.4; i.e.,

$$\tau' = \nu^{-k+j+1}\rho \otimes \nu^{\frac{-k}{2}+j+1}\zeta(\rho, k-1) \rtimes \zeta_1(\rho, \ell; \sigma),$$

$$\tau'' = \nu^{-j}\rho \otimes \nu^{\frac{-k}{2}+j}\zeta(\rho, k-1) \rtimes \zeta_1(\rho, \ell; \sigma),$$

$$\tau''' = \nu^{-\ell+1}\rho \otimes \nu^{\frac{-k+1}{2}+j}\zeta(\rho, k) \rtimes \zeta_1(\rho, \ell-1; \sigma).$$

Let

$$\tau_0' = \nu^{-k+j+1}\rho \otimes \nu^{\frac{-k}{2}+j+1}\zeta(\rho, k-1) \rtimes \zeta_1(\rho, \ell; \sigma_0),$$

$$\tau_0'' = \nu^{-j}\rho \otimes \nu^{\frac{-k}{2}+j}\zeta(\rho, k-1) \rtimes \zeta_1(\rho, \ell; \sigma_0),$$

$$\tau_0''' = \nu^{-\ell+1}\rho \otimes \nu^{\frac{-k+1}{2}+j}\zeta(\rho, k) \rtimes \zeta_1(\rho, \ell-1; \sigma_0).$$

Then, from $\sigma_0 \cong s\sigma_0$ and Lemmas 4.2 and 4.7, we can conclude

$$\tau_0' = r_{M^0,M}(\tau'),$$
$$\tau_0'' = r_{M^0,M}(\tau''),$$
$$\tau_0''' = r_{M^0,M}(\tau''')$$

and

$$s_{(m)}(\pi_0) = \tau_0' + \tau_0'' + \tau_0'''.$$

2. Suppose that $\sigma \cong \hat{s}\sigma$, i.e, $\sigma_0 \not\cong s\sigma_0$. Condition (C0) implies that this is possible only for m even. Then $\zeta_1(\rho, \ell; \sigma) \cong \hat{s}\zeta_1(\rho, \ell; \sigma)$, $\pi \cong \hat{s}\pi$ and $\tau \cong \hat{s}\tau$, for all $\tau \in T$. Theorem 3.4 for $SO(2n, F)$ follows from Lemma 4.5, (2) and Theorem 3.4 for $O(2n, F)$. According to Lemma 4.2,

$$r_{G^0,G}(\pi) = \pi_0 + s\pi_0.$$

We have

$$r_{M^0,G}(\pi) = r_{M^0,G^0}(\pi_0) + sr_{M^0,G^0}(\pi_0).$$

To find Jacquet modules of π_0, we select half of the components of restrictions of Jacquet modules of π. More precisely, we take the components containing σ_0 (not $s\sigma_0$).

Now, consider the case $r = 0$. Note that to have $(\rho, 1_0)$ satisfy (C0), we must have m even. By Lemma 4.6, $s\zeta_1(\rho, \ell; 1_0) \not\cong \zeta_1(\rho, \ell; 1_0)$. We apply the same reasoning as in the case $\sigma_0 \not\cong s\sigma_0$.

Propositions 3.2, 3.3 and Theorem 3.5 can be proved in a similar way. However, for Proposition 3.3, the case $r = 0$, m even has to be considered separately. We now give the proof of Proposition 3.3 for $r = 0$, m even. Let

$$\pi_0 = \nu^{\alpha}\zeta(\rho, k) \rtimes 1_0,$$
$$\pi = \nu^{\alpha}\zeta(\rho, k) \rtimes 1.$$

Then

$$\pi = i_{G,G^0}(\pi_0).$$

If π is irreducible, then π_0 is irreducible.

Suppose that π is reducible. Recall that (Lemma 4.7)

$$T_i(\rho; 1_0) \not\cong sT_i(\rho; 1_0),$$

so

$$T_i(\rho;1) \cong \hat{s}T_i(\rho;1)$$

and therefore

$$T_1(\rho;1) \not\cong \hat{s}T_2(\rho;1).$$

According to Lemma 4.1 or Lemma 4.7,

$$T_i(\rho;1) = i_{O(2m,F),SO(2m,F)}(T_i(\rho;1_0)).$$

By inspecting all the cases in Proposition 3.3, we see that there is one-to-one correspondence between components of π_0 and irreducible components of π.

Let ρ be an irreducible unitary supercuspidal representation of $GL(m,F)$, m odd. Suppose $\rho \cong \tilde{\rho}$. Let σ_0 be an irreducible supercuspidal representation of $SO(2n,F)$, $n \geq 0$. If $n > 0$, suppose that $\sigma_0 \not\cong s\sigma_0$.

Then, $\rho \rtimes \sigma_0$ is irreducible, but $\rho \rtimes \sigma$ is reducible (cf. Proposition 4.3).

Proposition 5.1. *Let ρ be an irreducible unitary supercuspidal representation of $GL(m,F)$, m odd. Suppose $\rho \cong \tilde{\rho}$. Let σ_0 be an irreducible supercuspidal representation of $SO(2n,F)$, $n \geq 0$. If $n > 0$, suppose that $\sigma_0 \not\cong s\sigma_0$. Let $\pi_0 = \nu^{\alpha}\rho \rtimes \zeta(\rho,\ell;\sigma_0)$ with $\alpha \in \mathbb{R}$, $\ell \geq 1$. Then, π_0 is reducible if and only if $\alpha \in \{\pm 1, \pm \ell\}$. Suppose π_0 is reducible. By contragredience, we may assume $\alpha \leq 0$.*

(1) $\alpha = -1$, $\ell = 1$
$\pi_0 = \pi_1^0 + \pi_2^0 + \pi_3^0 + \pi_4^0$ *with*

$$\pi_1^0 = L(\nu^{-1}\rho;\rho \rtimes \sigma_0), \quad \pi_2^0 = \delta(\nu\rho;\rho \rtimes \sigma_0), \quad \pi_3^0 = L(\nu^{-\frac{1}{2}}\delta(\rho,2);\sigma_0), \quad \pi_4^0 = s\pi_3^0.$$

In this case, π_1^0 is the unique irreducible subrepresentation, π_2^0 is the unique irreducible quotient, and $\pi_3^0 \oplus \pi_4^0$ is a subquotient. We have

$$s_{(m)}\pi_1^0 = \nu^{-1}\rho \otimes \rho \rtimes \sigma_0,$$

$$s_{(m)}\pi_2^0 = \nu\rho \otimes \rho \rtimes \sigma_0,$$

$$s_{(m)}\pi_3^0 = \rho \otimes L(\nu^{-1}\rho;\sigma_0).$$

(2) $\alpha = -1$, $\ell > 1$
$\pi_0 = \pi_1^0 + \pi_2^0$ *with*

$$\pi_1^0 = L([\nu^{-\ell+1}\rho,\nu^{-1}\rho],\nu^{-1}\rho;\rho \rtimes \sigma_0), \quad \pi_2^0 = L([\nu^{-\ell+1}\rho,\nu^{-1}\rho];\delta(\nu\rho;\rho \rtimes \sigma_0)).$$

In this case, π_1^0 is the unique irreducible subrepresentation and π_2^0 is the unique irreducible quotient. We have

(a) $\ell = 2$,

$$s_{(m)}\pi_1^0 = 2\nu^{-1}\rho \otimes L(\nu^{-1}\rho;\rho \rtimes \sigma_0) + \nu^{-1}\rho \otimes L(\nu^{-\frac{1}{2}}\delta(\rho,2);\sigma_0) + \nu^{-1}\rho \otimes s \cdot L(\nu^{-\frac{1}{2}}\delta(\rho,2);\sigma_0)$$

$$s_{(m)}\pi_2^0 = \nu^{-1}\rho \otimes \delta(\nu\rho;\rho \rtimes \sigma_0) + \nu\rho \otimes L(\nu^{-1}\rho;\rho \rtimes \sigma_0).$$

(b) $\ell > 2$,

$$s_{(m)}\pi_1^0 = \nu^{-\ell+1}\rho \otimes L([\nu^{-\ell+2}\rho,\nu^{-1}\rho],\nu^{-1}\rho;\rho \rtimes \sigma_0) + \nu^{-1}\rho \otimes L([\nu^{-\ell+1}\rho,\nu^{-1}\rho];\rho \rtimes \sigma_0)$$

$$s_{(m)}\pi_2^0 = \nu^{-\ell+1}\rho \otimes L([\nu^{-\ell+2}\rho,\nu^{-1}\rho];\delta(\nu\rho;\rho \rtimes \sigma_0)) + \nu\rho \otimes L([\nu^{-\ell+1}\rho,\nu^{-1}\rho];\rho \rtimes \sigma_0).$$

(3) $\alpha = -\ell$, $\ell > 1$

$$\pi_0 = \pi_1^0 + \pi_2^0 \text{ with}$$

$$\pi_1^0 = L([\nu^{-\ell}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0), \quad \pi_2^0 = L(\nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), [\nu^{-\ell+2}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0).$$

In this case, π_1^0 is the unique irreducible subrepresentation and π_2^0 is the unique irreducible quotient. We have

$$\begin{aligned} s_{(m)}\pi_1^0 &= \nu^{-\ell}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0), \\ s_{(m)}\pi_2^0 &= \nu^{-\ell+1}\rho \otimes L(\nu^{-\ell}\rho, [\nu^{-\ell+2}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0) \\ &\quad + \nu^{\ell}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0). \end{aligned}$$

Proof. First, we verify that π_0 is reducible if and only if $\alpha \in \{\pm 1, \pm \ell\}$. Let

$$\begin{aligned} \pi &= \nu^{\alpha}\rho \rtimes \zeta_1(\rho, \ell; \sigma), \\ \tau &= \nu^{\alpha}\rho \otimes \zeta_1(\rho, \ell; \sigma), \\ \tau_0 &= \nu^{\alpha}\rho \otimes \zeta(\rho, \ell; \sigma_0). \end{aligned}$$

We show that π is irreducible if and only if π_0 is irreducible. The result then follows from Proposition 3.2. Notice that $\tau_0 \cong s\tau_0$ and therefore $\pi_0 \cong s\pi_0$. By Lemmas 4.1 and 4.2,

$$\begin{aligned} i_{M,M^0}(\tau_0) &= \tau + \hat{s}\tau, \\ i_{G,G^0}(\pi_0) &= i_{G,M}(\tau + \hat{s}\tau) = \pi + \hat{s}\pi, \\ r_{G^0,G}(\pi) &= \pi_0. \end{aligned}$$

Therefore, if π_0 is irreducible, then π is irreducible. Conversely, suppose that π is irreducible. We can prove by Jacquet module considerations that $\hat{s}\pi \ncong \pi$. According to Lemma 4.1, $\pi_0 = r_{G^0,G}(\pi)$ is irreducible. Thus, we have the reducibility points claimed.

Now, suppose that π is reducible. We verify (1).

First, Proposition 3.2 tells us that π decomposes as $\pi = \pi_1 + \pi_2 + \pi_3$ where

$$\pi_1 = L(\nu^{-1}\rho; T_1(\rho; \sigma)), \quad \pi_2 = \delta(\nu\rho; T_1(\rho; \sigma)), \quad \pi_3 = L(\nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma).$$

Then

$$\pi_0 = r_{G^0,G}(\pi) = r_{G^0,G}(\pi_1 + \pi_2 + \pi_3) = \pi_1^0 + \pi_2^0 + \pi_3^0 + s\pi_3^0,$$

where

$$\pi_1^0 = L(\nu^{-1}\rho; \rho \rtimes \sigma_0), \quad \pi_2^0 = \delta(\nu\rho; \rho \rtimes \sigma_0), \quad \pi_3^0 = L(\nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0).$$

Note that $\pi_1^0 \cong s\pi_1^0$, $\pi_2^0 \cong s\pi_2^0$, $\pi_3^0 \ncong s\pi_3^0$.

Let N^0 be the standard Levi subgroup of $G^0 = SO(4m + 2n, F)$ isomorphic to $GL(m, F) \times SO(2m + 2n, F)$. Then

$$s_{(m)}\pi_1^0 = r_{N^0,G^0}(\pi_1^0) = r_{N^0,G^0} \circ r_{G^0,G}(\pi_1) = r_{N^0,N} \circ r_{N,G}(\pi_1) = r_{N^0,N}(s_{(m)}\pi_1).$$

Similarly, $s_{(m)}\pi_2^0 = r_{N^0,N}(s_{(m)}\pi_2)$. For π_3^0, we have

$$\begin{aligned} s_{(m)}\pi_3^0 + s \cdot s_{(m)}\pi_3^0 &= r_{N^0,G^0}(\pi_3^0 + s\pi_3^0) \\ &= r_{N^0,N} \circ r_{N,G}(\pi_3) \\ &= r_{N^0,N}(s_{(m)}\pi_3) \\ &= \rho \otimes L(\nu^{-1}\rho; \sigma_0) + \rho \otimes sL(\nu^{-1}\rho; \sigma_0). \end{aligned}$$

Let M^0 be the standard Levi subgroup of $G^0 = SO(4m+2n, F)$ isomorphic to $GL(m,F) \times GL(m,F) \times SO(2m+2n,F)$. From the Langlands data, we see that

$$\begin{gathered} r_{N^0,G^0}(\pi_3) \geq \nu^{-\frac{1}{2}}\delta(\rho,2) \otimes \sigma_0 \\ \Downarrow \\ r_{M^0,G^0}(\pi_3) \geq \rho \otimes \nu^{-1}\rho \otimes \sigma_0. \end{gathered}$$

It follows that

$$s_{(m)}\pi_3^0 = \rho \otimes L(\nu^{-1}\rho; \sigma_0).$$

Proposition 3.2 and the exactness of the functor $r_{G^0,G}$ tell us that π_1^0 is a subrepresentation, π_2^0 a quotient, and $\pi_3^0 \oplus \pi_4^0$ a subquotient of π_0. To see that π_1^0 is the unique irreducible subrepresentation, suppose that π_i^0 is a subrepresentation of π_0. Applying i_{G,G^0} to the exact sequence

$$0 \longrightarrow \pi_i^0 \longrightarrow \pi_0 \longrightarrow \pi_0/\pi_i^0 \longrightarrow 0,$$

we obtain

$$0 \longrightarrow i_{G,G^0}(\pi_i^0) \longrightarrow \pi \oplus \hat{s}\pi \longrightarrow i_{G,G^0}(\pi_0/\pi_i^0) \longrightarrow 0.$$

It follows that $i_{G,G^0}(\pi_i^0)$ is a subrepresentation of $\pi \oplus \hat{s}\pi$. Since $\pi_i \hookrightarrow i_{G,G^0}(\pi_i^0)$, we have $\pi_i \hookrightarrow \pi \oplus \hat{s}\pi$. Therefore, $\pi_i \cong \pi_1$ or $\pi_i \cong \hat{s}\pi_1$. By checking all irreducible subquotients of π, we obtain $\pi_i \cong \pi_1$. It follows that π_1^0 is the unique subrepresentation of π_0. The proof that π_2^0 is the unique quotient of π_0 is similar.

(2) and (3) are done analogously. In these cases, the composition series for π_0 follow from Lemma 4.5. □

Proposition 5.2. *Let ρ be an irreducible unitary supercuspidal representation of $GL(m,F)$, m odd. Suppose $\rho \cong \tilde{\rho}$. Let σ_0 be an irreducible supercuspidal representation of $SO(2n,F)$, $n \geq 0$. If $n > 0$, suppose that $\sigma_0 \ncong s\sigma_0$. Let $\pi_0 = \nu^{\alpha}\zeta(\rho,k) \rtimes \sigma_0$ with $\alpha \in \mathbb{R}$, $k \geq 2$. Then π_0 is reducible if and only if $\alpha \in \{\frac{-k+1}{2}, \frac{-k+3}{2}, \ldots, \frac{k-1}{2}\} \setminus \{0\}$. Suppose π_0 is reducible. By contragredience, we may assume that $\alpha \leq 0$. Write $\alpha = \frac{-k+1}{2} + j$ with $0 \leq j \leq \frac{k-3}{2}$. Then $\pi_0 = \pi_1^0 + \pi_2^0$ with*

$$\pi_1^0 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0)$$

and

$$\pi_2^0 = s^{j+1}L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \nu^{-j+\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma_0).$$

In this case, π_1^0 is the unique irreducible subrepresentation and π_2^0 the unique irreducible quotient.

(1) *$j = 0 = \frac{k-2}{2}$ $(k=2)$,*

$$\begin{aligned} s_{(m)}\pi_1^0 &= \nu^{-1}\rho \otimes (\rho \rtimes \sigma_0), \\ s_{(m)}\pi_2^0 &= \rho \otimes sL(\nu^{-1}\rho; \sigma_0). \end{aligned}$$

(2) *$j = 0$, $k > 2$,*

$$\begin{aligned} s_{(m)}\pi_1^0 &= \nu^{-k+1}\rho \otimes L([\nu^{-k+2}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0), \\ s_{(m)}\pi_2^0 &= \nu^{-k+1}\rho \otimes sL([\nu^{-k+2}\rho, \nu^{-2}\rho], \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma_0) \\ &\quad + \rho \otimes sL([\nu^{-k+1}\rho, \nu^{-1}\rho]; \sigma_0). \end{aligned}$$

(3) $j = \frac{k-2}{2}$, $k \geq 4$ (k *even*),

$$\begin{aligned} s_{(m)}\pi_1^0 &= \nu^{-\frac{k}{2}}\rho \otimes L([\nu^{-\frac{k}{2}+1}\rho, \nu^{-1}\rho], [\nu^{-\frac{k}{2}+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0) \\ &+ \nu^{-\frac{k}{2}+1}\rho \otimes sL([\nu^{-\frac{k}{2}}\rho, \nu^{-1}\rho], [\nu^{-\frac{k}{2}+2}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0) \end{aligned}$$

$$s_{(m)}\pi_2^0 = \nu^{-\frac{k}{2}+1}\rho \otimes s^{j+1}L(\nu^{-\frac{k}{2}}\rho, \nu^{\frac{-k+3}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma_0).$$

(4) $0 < j < \frac{k-2}{2}$,

$$\begin{aligned} s_{(m)}\pi_1^0 &= \nu^{-k+j+1}\rho \otimes L([\nu^{-k+j+2}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0) \\ &+ \nu^{-j}\rho \otimes sL([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0), \end{aligned}$$

$$\begin{aligned} s_{(m)}\pi_2^0 &= \nu^{-k+j+1}\rho \otimes s^{j+1}L([\nu^{-k+j+2}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma_0) \\ &+ \nu^{-j}\rho \otimes s^{j+1}L([\nu^{-k+j+1}\rho, \nu^{-j-1}\rho], \nu^{-j+\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma_0). \end{aligned}$$

Proof. Let $\pi = \nu^{\alpha}\zeta(\rho,k) \rtimes \sigma$. Then $\pi = i_{G,G^0}(\pi_0)$. We consider the cases from Proposition 3.3.

(i) If $j = \frac{k-1}{2}$ (i.e., $\alpha = 0$), then $\pi = \pi_1 + \pi_2$ with

$$\pi_i = L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho]; T_i(\rho;\tau))$$

for $i = 1, 2$. We have

$$\begin{aligned} \pi_0 + s\pi_0 &= r_{G^0,G} \circ i_{G,G^0}(\pi_0) = r_{G^0,G}(\pi) \\ &= r_{G^0,G}(\pi_1 + \pi_2) = 2L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0). \end{aligned}$$

It follows that

$$\pi_0 = L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0)$$

is irreducible.

(ii) Let $0 \leq j < \frac{k-1}{2}$. By Proposition 3.3, $\pi = \pi_1 + \pi_2 + \pi_3$ with

$$\pi_i = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; T_i(\rho;\sigma))$$

for $i = 1, 2$ and

$$\pi_3 = L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \nu^{-j+\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma).$$

We have

$$\begin{aligned} \pi_0 + s\pi_0 &= r_{G^0,G} \circ i_{G,G^0}(\pi_0) = r_{G^0,G}(\pi) = r_{G^0,G}(\pi_1 + \pi_2 + \pi_3) \\ &= 2L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0) \\ &+ L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \nu^{-j+\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma_0) \\ &+ sL([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \nu^{-j+\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma_0). \end{aligned}$$

It follows that $\pi_0 = \pi_1^0 + \pi_2^0$ with

$$\pi_1^0 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0)$$

and

$$\pi_2^0 = s^{\epsilon}L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \nu^{-j+\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma_0),$$

where $\epsilon = 0$ or 1. Consider the standard Levi subgroup of G^0,

$$Q^0 \cong \underbrace{GL(m,F) \times \cdots \times GL(m,F)}_{k} \times SO(2n,F).$$

Let

$$\psi = \nu^{-k+j+1}\rho \otimes \nu^{-j-2}\rho \otimes (\nu^{-j}\rho \otimes \nu^{-j-1}\rho) \otimes (\nu^{-j+1}\rho \otimes \nu^{-j}\rho) \otimes \cdots \otimes (\nu^{-1}\rho \otimes \nu^{-2}\rho) \otimes (\rho \otimes \nu^{-1}\rho) \otimes \sigma_0.$$

By Frobenius reciprocity,

$$r_{Q^0,G^0}(L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \nu^{-j+\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma_0)) \geq \psi$$

and

$$r_{s(Q^0),G^0}(sL([\nu^{-k+j+1}\rho,\nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \nu^{-j+\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma_0)) \geq s\psi.$$

We apply Corollary 5.3 and Lemma 6.2 of [Ban3] to compute $r_{Q^0,G^0}(\pi_0)$. We observe that, for j odd, the multiplicity of ψ in $r_{Q^0,G^0}(\pi_0)$ is 1 and the multiplicity of $s\psi$ in $r_{s(Q^0),G^0}(\pi_0)$ is 0. It follows that $\epsilon = 0$. Similarly, for j even, the multiplicity ψ in $r_{Q^0,G^0}(\pi_0)$ is 0 and the multiplicity of $s\psi$ in $r_{s(Q^0),G^0}(\pi_0)$ is 1. It follows that $\epsilon = 1$.

Now,

$$s_{(m)}\pi_0 + s \cdot s_{(m)}\pi_0 = r_{N^0,G^0}(\pi_0 + s\pi_0) = r_{N^0,G}(\pi) = r_{N^0,N}(s_{(m)}\pi_1 + s_{(m)}\pi_2 + s_{(p)}\pi_3).$$

Consider the case (a) from the Proposition 3.3. Then

$$s_{(m)}\pi_0 + s \cdot s_{(m)}\pi_0 = 2\nu^{-1}\rho \otimes (\rho \rtimes \sigma_0) + \rho \otimes L(\nu^{-1}\rho; \sigma_0) + \rho \otimes sL(\nu^{-1}\rho; \sigma_0).$$

It follows that

$$s_{(m)}\pi_0 = \nu^{-1}\rho \otimes (\rho \rtimes \sigma_0) + \rho \otimes L(\nu^{-1}\rho; \sigma_0)$$

and

$$s_{(m)}\pi_1^0 = \nu^{-1}\rho \otimes (\rho \rtimes \sigma_0),$$
$$s_{(m)}\pi_2^0 = \rho \otimes s^{\epsilon} L(\nu^{-1}\rho; \sigma_0),$$

where $\epsilon = 0$ or 1.

To determine ϵ, observe that in general,

$$s_{(m)}(\pi) = \nu^{\alpha + \frac{-k+1}{2}}\rho \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho, k-1) \rtimes \sigma + \nu^{-\alpha+\frac{-k+1}{2}}\tilde{\rho} \otimes \nu^{\alpha-\frac{1}{2}}\zeta(\rho, k-1) \rtimes \sigma.$$

Since m is odd, an odd number of sign changes are required to produce $\nu^{-\alpha+\frac{-k+1}{2}}\tilde{\rho}$. Therefore,

$$s_{(m)}(\pi_0) = \nu^{\alpha+\frac{-k+1}{2}}\rho \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho, k-1) \rtimes \sigma_0 + \nu^{-\alpha+\frac{-k+1}{2}}\tilde{\rho} \otimes s(\nu^{\alpha-\frac{1}{2}}\zeta(\rho, k-1) \rtimes \sigma_0).$$

When $j = 0 = \frac{k-2}{2}$ $(k = 2)$, we get $\alpha = -\frac{1}{2}$ and

$$\begin{aligned} s_{(m)}(\pi_0) &= \nu^{-1}\rho \otimes (\rho \rtimes \sigma_0) + \rho \otimes s(\nu^{-1}\rho \rtimes \sigma_0) \\ &= \nu^{-1}\rho \otimes (\rho \rtimes \sigma_0) + \rho \otimes sL(\nu^{-1}\rho; \sigma_0). \end{aligned}$$

Thus, we see that $\epsilon = 1$.

For the remaining cases, the proofs are similar. □

Theorem 5.3. *Let ρ be an irreducible unitary supercuspidal representation of $GL(m, F)$, m odd. Suppose $\rho \cong \tilde{\rho}$. Let σ_0 be an irreducible supercuspidal representation of $SO(2n, F)$, $n \geq 0$. If $n > 0$, suppose that $\sigma_0 \not\cong s\sigma_0$. Let $\pi_0 =$*

$\nu^\alpha\zeta(\rho,k) \rtimes \zeta(\rho,\ell;\sigma_0)$. Suppose $k \geq 2$ and $\ell \geq 1$ (the cases $k=1$ and $\ell=0$ are covered by Propositions 5.1 and 5.2 above). Then, π_0 is reducible if and only if

$$\alpha \in \quad \{\pm(\ell+\tfrac{k-1}{2}), \pm(\ell+\tfrac{k-1}{2}-1), \dots, \pm(\ell+\tfrac{-k+1}{2})\}$$
$$\cup\{\{\tfrac{-k-1}{2}, \tfrac{-k-1}{2}+1, \dots, \tfrac{k+1}{2}\} \setminus \{0 \text{ if } k=2\ell-1\}\}.$$

(We note that these sets need not be disjoint.) Let S_1 denote the first set, and S_2 the second. Suppose π_0 is reducible. By contragredience, we may restrict our attention to the case $\alpha \leq 0$.

(1) $\alpha \notin S_2$.

In this case, we have $\pi_0 = \pi_1^0 + \pi_2^0$, where

$$\pi_1^0 = L([\nu^{\alpha-\frac{k-1}{2}}\rho, \nu^{\alpha+\frac{k-1}{2}}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0),$$
$$\pi_2^0 = L([\nu^{\alpha-\frac{k-1}{2}}\rho, \nu^{-\ell-1}\rho],$$
$$\nu^{-\ell+\frac{1}{2}}\delta(\rho,2), \nu^{-\ell+\frac{3}{2}}\delta(\rho,2), \dots, \nu^{\alpha+\frac{k}{2}}\delta(\rho,2), [\nu^{\alpha+\frac{k-1}{2}+2}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0).$$

π_1^0 is the unique irreducible subrepresentation and π_2^0 is the unique irreducible quotient.

(2) $\alpha = \frac{-k-1}{2}$.

One component of π_0 is the following:

$$\pi_1^0 = L([\nu^{-k}\rho, \nu^{-1}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0).$$

The other components are described below.

(a) $\ell = 1$ *(so $k > \ell - 1$).*

In this case, there are three additional components:

$$\pi_2^0 = L([\nu^{-k}\rho, \nu^{-2}\rho]; \delta(\nu\rho; \rho \rtimes \sigma_0)),$$
$$\pi_3^0 = L([\nu^{-k}\rho, \nu^{-2}\rho], \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma_0)$$

and

$$\pi_4^0 = s\pi_3^0.$$

π_1^0 is the unique irreducible subrepresentation, π_2^0 is the unique irreducible quotient, and $\pi_3^0 \oplus \pi_4^0$ is a subquotient.

(b) $k > \ell - 1 > 0$.

In this case, there are four additional components:

$$\pi_2^0 = L([\nu^{-k}\rho, \nu^{-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \delta(\nu\rho; \rho \rtimes \sigma_0)),$$
$$\pi_3^0 = L([\nu^{-k}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho,2), \nu^{-\ell+\frac{3}{2}}\delta(\rho,2), \dots, \nu^{-\frac{3}{2}}\delta(\rho,2); \delta(\nu\rho; \rho \rtimes \sigma_0))$$
$$\pi_4^0 = L([\nu^{-k}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho,2), \nu^{-\ell+\frac{3}{2}}\delta(\rho,2), \dots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma_0)$$
$$\pi_5^0 = s\pi_4^0.$$

π_1^0 is the unique irreducible subrepresentation, π_3^0 is the unique irreducible quotient, and $\pi_2^0 \oplus \pi_4^0 \oplus \pi_5^0$ is a subquotient.

(c) $\ell - 1 = k$.

In this case, there is one additional component:

$$\pi_2^0 = L([\nu^{-k}\rho, \nu^{-2}\rho], [\nu^{-k}\rho, \nu^{-1}\rho]; \delta(\nu\rho; \rho \rtimes \sigma_0)).$$

π_1^0 is the unique irreducible subrepresentation and π_2^0 is the unique irreducible quotient.

(d) $\ell - 1 > k$.
In this case, there is one additional component:

$$\pi_2^0 = L([\nu^{-\ell+1}\rho, \nu^{-2}\rho], [\nu^{-k}\rho, \nu^{-1}\rho]; \delta(\nu\rho; \rho \rtimes \sigma_0)).$$

π_1^0 is the unique irreducible subrepresentation and π_2^0 is the unique irreducible quotient.

(3) $\alpha \in S_2$.
Write $\alpha = \frac{-k+1}{2} + j$, with $0 \le j \le \frac{k-1}{2}$. One component of π_0 is π_1^0, where π_1^0 is defined as follows:

$$\pi_1^0 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \times \rho \rtimes \sigma_0).$$

The remaining components are described below, on a case by case basis.

(a) $k - j - 1 > j > \ell - 1$.
We have two additional components:

$$\pi_2^0 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \rho \rtimes \sigma_0),$$

$$\pi_3^0 = L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \rho \rtimes \sigma_0).$$

π_3^0 is the unique irreducible quotient and $\pi_1^0 \oplus \pi_2^0$ is a subrepresentation.

(b) $k - j - 1 = j > \ell - 1$.
We have one additional component:

$$\pi_2^0 = L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \rho \rtimes \sigma_0).$$

In this case, $\pi_0 = \pi_1^0 \oplus \pi_2^0$.

(c) $k - j - 1 > j = \ell - 1$.
We have one additional component:

$$\pi_2^0 = L([\nu^{-k+j+1}\rho, \nu^{-\ell-1}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \rho \rtimes \sigma_0).$$

π_1^0 is the unique irreducible subrepresentation and π_2^0 is the unique irreducible quotient.

(d) $k - j - 1 > \ell - 1 > j$.
We have three additional components:

$$\pi_2^0 = L([\nu^{-k+j+1}\rho, \nu^{-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \delta(\nu\rho; \rho \rtimes \sigma_0)),$$

$$\pi_3^0 = L([\nu^{-k+j+1}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-j-\frac{5}{2}}\delta(\rho, 2), \nu^{-j-1}\delta(\rho, 3), \nu^{-j}\delta(\rho, 3), \dots, \nu^{-1}\delta(\rho, 3); \delta(\nu\rho; \rho \rtimes \sigma_0)).$$

$$\pi_4^0 = L([\nu^{-k+j+1}\rho, \nu^{-\ell-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \rho \rtimes \sigma_0).$$

π_1^0 is the unique irreducible subrepresentation, π_3^0 is the unique irreducible quotient, and $\pi_2^0 \oplus \pi_4^0$ is a subquotient.

(e) $k - j - 1 = \ell - 1 > j$.
We have one additional component:

$$\pi_2^0 = L([\nu^{-\ell+1}\rho, \nu^{-k+\ell-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-2}\rho], \nu^{-k+\ell-\frac{1}{2}}\delta(\rho, 2), \nu^{-k+\ell+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \delta(\nu\rho; \rho \rtimes \sigma_0)).$$

π_1^0 is the unique irreducible subrepresentation and π_2^0 is the unique irreducible quotient.

(f) $\ell - 1 > k - j - 1 > j$.

(i) *If $j = 0$, the representation π_2^0 below is the only other component. In this case, π_1^0 is the unique irreducible subrepresentation and π_2^0 is the unique irreducible quotient.*

(ii) *If $j > 0$, there are two additional components:*

$$\pi_2^0 = L([\nu^{-k+j+1}\rho, \nu^{-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \nu^{-j+\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \delta(\nu\rho; \rho \rtimes \sigma_0)),$$

$$\pi_3^0 = L([\nu^{-\ell+1}\rho, \nu^{-k+j-1}\rho], [\nu^{-j}\rho, \nu^{-2}\rho], \nu^{-k+j+\frac{1}{2}}\delta(\rho,2), \nu^{-k+j+\frac{3}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \delta(\nu\rho; \rho \rtimes \sigma_0)).$$

In this case, π_2^0 is the unique irreducible quotient and $\pi_1^0 \oplus \pi_3^0$ is a subrepresentation.

(g) $\ell - 1 > k - j - 1 = j$.

We have one additional component:

$$\pi_2^0 = L([\nu^{-\ell+1}\rho, \nu^{\frac{-k-3}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-2}\rho], \nu^{-\frac{k}{2}}\delta(\rho,2), \nu^{-\frac{k}{2}+1}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \delta(\nu\rho; \rho \rtimes \sigma_0)).$$

In this case, $\pi_0 = \pi_1^0 \oplus \pi_2^0$.

We note that the case $k - j - 1 = j = \ell - 1$ is a point of irreducibility.

Proof. Let

$$\pi = \nu^{\alpha}\zeta(\rho, k) \rtimes \zeta_1(\rho, \ell; \sigma).$$

As in the proof of Proposition 5.1, we obtain

$$\pi_0 = r_{G^0,G}(\pi).$$

The irreducible subquotients and composition series structure of π are described in Theorem 3.4. We argue as above to obtain the corresponding results for π_0. □

Remark 5.4. Let π_0, π be as in the proof of Theorem 5.3. The proof of Theorem 3.4 describes Jacquet modules of components of π. We may apply restriction to obtain Jacquet modules of irreducible components of π_0, as follows: Let π_i be a component of π. If $r_{G^0,G}(\pi_i) = \pi_i^0$ is irreducible, then

$$s_{(m)}(\pi_i^0) = r_{G^0,G}(s_{(m)}(\pi_i)).$$

If $r_{G^0,G}(\pi_i)$ is reducible, then $r_{G^0,G}(\pi_i) = \pi_i^0 + s \cdot \pi_i^0$ and

$$r_{G^0,G}(s_{(m)}(\pi_i)) = s_{(m)}(\pi_i^0) + s \cdot s_{(m)}(\pi_i^0). \tag{1}$$

The components of $s_{(m)}(\pi_i^0)$ can be computed from (1) up to s. We can determine whether s appears in a component of $s_{(m)}(\pi_i^0)$ by observing that if $r_{G^0,G}(\pi_i)$ is reducible, then $\pi_i^0 \not\cong s\pi_i^0$. Such components appear only in cases 2(a) and (b) (in particular, π_3^0, π_4^0 for 2(a) and π_4^0, π_5^0 for 2(b)). In either case, exactly one of $\pi_i^0, s\pi_i^0$ appears as a component of $\nu^{\frac{-k+\ell-1}{2}}\zeta(\rho, k+\ell) \rtimes \sigma_0$, hence has Jacquet module described in Proposition 5.2.

Suppose $\rho \not\cong \rho_0$ are representations of $GL(m,F)$ and $GL(m_0,F)$. We continue to assume $\rho \cong \tilde{\rho}$ with m odd and either $\sigma_0 \not\cong s\sigma_0$ or $\sigma_0 = 1_0$. In order to build on Theorem 3.5, we want (ρ_0, σ) to satisfy (C0). We can characterize this in terms of

ρ_0 and σ_0 by requiring that either (ρ_0, σ_0) satisfies (C0) or $\rho_0 \cong \tilde{\rho}_0$ with m_0 odd (by Proposition 4.3). Then, by [Gol1], $\rho_0 \times \rho \rtimes \sigma_0$ is the direct sum of two irreducible subrepresentations. Write

$$\rho_0 \times \rho \rtimes \sigma_0 = T_1(\rho_0, \rho; \sigma_0) + T_2(\rho_0, \rho; \sigma_0).$$

Theorem 5.5. *Suppose $\rho \ncong \rho_0$ are irreducible unitary supercuspidal representations of $GL(m, F)$ and $GL(m_0, F)$. Assume that $\rho \cong \tilde{\rho}$ and m odd. Let σ_0 be an irreducible supercuspidal representation of $SO(2n, F)$, $n \geq 0$. If $n > 0$, suppose that $\sigma_0 \ncong s\sigma_0$. We also assume that either (ρ_0, σ_0) satisfies (C0) or $\rho_0 \cong \tilde{\rho}_0$ with m_0 odd. Let $\pi_0 = \nu^\alpha \zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma_0)$ with $\alpha \in \mathbb{R}$, $k \geq 1$. Then π_0 is reducible if and only if $\alpha \in \{\frac{-k+1}{2}, \frac{-k+3}{2}, \ldots, \frac{k-1}{2}\}$. Suppose π_0 is reducible. By contragredience, we may assume that $\alpha \leq 0$. Write $\alpha = \frac{-k+1}{2} + j$ with $0 \leq j \leq \frac{k-1}{2}$.*

(1) $j = \frac{k-1}{2}$

$\pi_0 = \pi_1^0 + \pi_2^0$ *with*

$$\pi_i^0 = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-1}\rho_0], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-1}\rho_0]; T_i(\rho_0, \rho; \sigma_0))$$

for $i = 1, 2$. In this case, $\pi_0 = \pi_1^0 \oplus \pi_2^0$.

(2) $0 \leq j < \frac{k-1}{2}$

$\pi_0 = \pi_1^0 + \pi_2^0 + \pi_3^0$ *with*

$$\pi_i^0 = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{-k+j+1}\rho_0, \nu^{-1}\rho_0], [\nu^{-j}\rho_0, \nu^{-1}\rho_0]; T_i(\rho_0, \rho; \sigma_0))$$

for $i = 1, 2$ and

$$\pi_3^0 = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{-k+j+1}\rho_0, \nu^{-j-2}\rho_0], \nu^{-j-\frac{1}{2}}\delta(\rho_0, 2), \nu^{-j+\frac{1}{2}}\delta(\rho_0, 2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho_0, 2); \rho \rtimes \sigma_0).$$

In this case, π_3^0 is the unique irreducible quotient and $\pi_1^0 \oplus \pi_2^0$ is a subrepresentation.

Proof. Let $\pi = \nu^\alpha \zeta(\rho_0, k) \rtimes \zeta_1(\rho, \ell; \sigma)$. The irreducible subquotients and composition series structure of π are described in Theorem 3.4. We argue as above to obtain the corresponding results for π_0. The Jacquet modules may be determined as in (the first part of) Remark 5.4. □

Remark 5.6. Let ρ and σ be as in the preceding corollary. Suppose ρ_0 is an irreducible unitary supercuspidal representation of $GL(m_0, F)$ with $\rho_0 \ncong \tilde{\rho}_0$. Then, $\nu^\alpha \zeta(\rho_0, k) \rtimes \zeta_1(\rho, \ell; \sigma)$ is irreducible for all $\alpha \in \mathbb{R}$.

6. Appendix

Let G be the group of F-points of a quasi-split reductive algebraic group defined over F. Let G^0 denote the connected component of the identity in G. For convenience, suppose that

$$G = G^0 \rtimes C,$$

with C a finite abelian group (that G is the semidirect product of G^0 and C is not required, but it is easier to formulate in that case). We now describe how to get from the general formulation of the Langlands classification ([B-W], [Sil1], [B-J1]) to the explicit description for even-orthogonal groups given in section 2 (patterned after [Tad2], [Jan2]).

We recall that we call an irreducible representation of G tempered if its restriction to G^0 is tempered (cf. Definition 2.5, [B-J1]).

Let Π be the set of simple roots for G^0. For $\Phi \subset \Pi$, we let $P_\Phi = M_\Phi U_\Phi$ be the standard parabolic subgroup of G^0 determined by Φ. Fix an order on Π. Then, there is a lexicographic order on subsets of Π. We define

$$X_C = \{\Phi \subset \Pi \,|\, \Phi \text{ is maximal among } \{c \cdot \Phi\}_{c \in C}\}.$$

Let $C(\Phi) = \{c \in C \,|\, c \cdot \Phi = \Phi\}$. We call $P = MU_\Phi$, where $M_\Phi \leq M \leq M_\Phi \rtimes C(\Phi)$ and $\Phi \in X_C$, a standard parabolic subgroup of G.

Set $P^0 = P_\Phi$. Let A be the split component of M_Φ, $\mathfrak{a}$ the real Lie algebra of A, and $\mathfrak{a}^*$ its dual. Let $\Pi(P^0, A) \subset \mathfrak{a}^*$ denote the set of simple roots corresponding to the pair (P^0, A). We set

$$\mathfrak{a}^*_- = \{x \in \mathfrak{a}^* \,|\, \langle x, \alpha \rangle < 0, \forall \alpha \in \Pi(P^0, A)\},$$
$$\mathfrak{a}^*_-(C) = \{x \in \mathfrak{a}^*_- \,|\, x \succeq c \cdot x, \forall c \in C(\Phi)\},$$

where $\langle \cdot, \cdot \rangle$ is a $C(\Phi)$-invariant inner product on $\mathfrak{a}^* \times \mathfrak{a}^*$ and $\succeq$ is the lexicographic order inherited from the order on Π (cf. section 3, [B-J1] for details).

Definition 6.1. A **set of Langlands data** for G is a triple (P, x, τ) with the following properties:

(1) $P = MU$ is a standard parabolic subgroup of G.
(2) $x \in \mathfrak{a}^*_-(C)$.
(3) $M = M_\Phi \rtimes C(\Phi, x)$, where $C(\Phi, x) = \{c \in C(\Phi) \,|\, c \cdot x = x\}$.
(4) $\tau \in Irr(M)$ is tempered.

Theorem 6.2 (The Langlands classification)**.** *There is a bijective correspondence*

$$Lang(G) \longleftrightarrow Irr(G),$$

where $Lang(G)$ denotes the set of all triples of Langlands data. Furthermore, if $(P, x, \tau) \leftrightarrow \pi$ under this correspondence, then π is the unique irreducible subrepresentation of $i_{G,M}(exp\, x \otimes \tau)$.

We now consider the group $SO(2n, F)$. The maximal split torus $A_\emptyset$ of $SO(2n, F)$ is

$$A_\emptyset = \left\{diag(a_1, \ldots, a_n, a_n^{-1}, \ldots, a_1^{-1}) \mid a_i \in F^\times\right\} \cong (F^\times)^n.$$

Let a denote the usual isomorphism of $(F^\times)^n$ into $A_\emptyset$, defined by

$$a(a_1, ..., a_n) = diag(a_1, \ldots, a_n, a_n^{-1}, \ldots, a_1^{-1}).$$

The group $X(A_\emptyset)_F$ of F-rational characters of $A_\emptyset$ has a basis $\left\{e_1^0, \ldots, e_n^0\right\}$, where e_i^0 is defined by

$$e_i^0(a(a_1, ..., a_n)) = a_i.$$

Thus, $\mathfrak{a}_0^* = \left\{x_1 e_1^0 + \cdots + x_n e_n^0 \mid x_i \in \mathbb{R}\right\}$. The roots of $SO(2n, F)$ form a root system of type D_n. The set of simple roots is $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i = e_i^0 - e_{i-1}^0$, for $1 \leq i \leq n-1$, and $\alpha_n = e_{n-1}^0 + e_n^0$. For our order on Π, we take $\alpha_i > \alpha_j$ if $i < j$.

Let $\Phi \subset \Pi$. We now describe the standard parabolic subgroup $P_\Phi = M_\Phi U_\Phi$. Write Φ in the form $\Phi = \Pi \backslash \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$, where $i_1 < i_2 < ... < i_k$. We have two cases.

A) If $\alpha_{n-1} \in \Phi$ or $\alpha_{n-1} \notin \Phi$, $\alpha_n \notin \Phi$, then

$$M_\Phi = \left\{ diag(g_1, ..., g_k, h, {}^\tau g_k^{-1}, ..., {}^\tau g_1^{-1}) \mid g_i \in GL(n_i, F),\ h \in SO(2(n-m), F) \right\},$$

where $n_1 = i_1, n_1 + n_2 = i_2, ..., n_1 + \cdots + n_k = i_k = m$. We have

$$M_\Phi \cong GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times SO(2(n-m), F).$$

B) If $\alpha_{n-1} \notin \Phi$, $\alpha_n \in \Phi$, then

$$M_\Phi = sM_{\Phi'}s,$$

where $\Phi' = s(\Phi)$,

$$M_\Phi = \left\{ diag(g_1, ..., g_k, {}^\tau g_k^{-1}, ..., {}^\tau g_1^{-1}) \mid g_i \in GL(n_i, F) \right\}.$$

Now, we will describe the set $\mathfrak{a}_-^*$ which appears in the definition of Langlands data. We have four cases.

A) Suppose that $\alpha_{n-1} \in \Phi$ or $\alpha_{n-1} \notin \Phi$, $\alpha_n \notin \Phi$. Then

$$A = \{a(\underbrace{a_1, \ldots, a_1}_{n_1}, \ldots, \underbrace{a_k, \ldots, a_k}_{n_k}, \underbrace{1, \ldots, 1}_{n-m}) \mid a_i \in F^\times\}.$$

The basis for $X(A)_F$ is $\{e_1, \ldots, e_k\}$, where

$$e_j : a(\underbrace{a_1, \ldots, a_1}_{n_1}, \ldots, \underbrace{a_k, \ldots, a_k}_{n_k}, \underbrace{1, \ldots, 1}_{n-m}) \mapsto a_j.$$

We have $\mathfrak{a}^* = \{x_1e_1 + \cdots + x_ke_k \mid x_j \in \mathbb{R}\}$. The set $\Pi(P, A) = \{\beta_1, \ldots, \beta_k\} \subset \mathfrak{a}^*$ corresponds to $\Pi \setminus \Phi = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\} \subset \mathfrak{a}_0^*$. For $1 \le j \le k-1$, $\beta_j = e_{j-1} - e_j$.

A.1) If $\alpha_{n-1} \in \Phi$, $\alpha_n \in \Phi$, then $\beta_k = e_k$ ($\alpha_{i_k} = \alpha_m = e_m^0 - e_{m+1}^0$). Take $x = x_1e_1 + \cdots + x_ke_k \in \mathfrak{a}^*$. Then

$$x \in \mathfrak{a}_-^* \Leftrightarrow \begin{cases} x_1 - x_2 < 0, \\ \vdots \\ x_{k-1} - x_k < 0, \\ x_k < 0. \end{cases}$$

This implies $x_1 < \cdots < x_k < 0$, so

$$\mathfrak{a}_-^* = \{x_1e_1 + \cdots + x_ke_k \mid x_1 < \cdots < x_k < 0\}.$$

A.2) If $\alpha_{n-1} \in \Phi$, $\alpha_n \notin \Phi$, then $\beta_k = 2e_k$ ($\alpha_{i_k} = \alpha_n = e_{n-1}^0 + e_n^0$), and

$$\mathfrak{a}_-^* = \{x_1e_1 + \cdots + x_ke_k \mid x_1 < \cdots < x_k < 0\}.$$

A.3) If $\alpha_{n-1} \notin \Phi$, $\alpha_n \notin \Phi$, then $n_k = 1$, $\beta_k = e_{k-1} + e_k$ and

$$\mathfrak{a}_-^* = \{x_1e_1 + \cdots + x_ke_k \mid x_1 < \cdots < x_{k-1} < -|x_k|\}.$$

B) Suppose that $\alpha_{n-1} \notin \Phi$, $\alpha_n \in \Phi$. Then

$$A = \{a(\underbrace{a_1, \ldots, a_1}_{n_1}, \ldots, \underbrace{a_k, \ldots, a_k}_{n_k - 1}, a_k^{-1}) \mid a_i \in F^\times\},$$

and we have $\beta_k = 2e_k$ ($\alpha_{i_k} = \alpha_n = e_{n-1}^0 - e_n^0$),

$$\mathfrak{a}_-^* = \{x_1e_1 + \cdots + x_ke_k \mid x_1 < \cdots < x_k < 0\}.$$

For $x = x_1e_1 + \cdots + x_ke_k \in \mathfrak{a}^*$, we have

$$exp\, x = \nu^{x_1} \otimes \cdots \otimes \nu^{x_k} \otimes 1.$$

This means that the value of $exp\, x$ on $m = diag(g_1, ..., g_k, h, {}^\tau g_k^{-1}, ..., {}^\tau g_1^{-1})$ is

$$exp\, x(m) = |det\, g_1|^{x_1} \dots |det\, g_k|^{x_k}.$$

Proposition 6.3 (The Langlands classification for $SO(2n, F)$)**.** *(i) Let ρ_i, $i = 1, \dots, k$, be an irreducible essentially tempered representation of $GL(n_i, F)$ and τ_0 an irreducible tempered representation of $SO(2m, F)$.*

(1) *Suppose that $m \geq 1$ and $e(\rho_1) < \cdots < e(\rho_k) < 0$. Then the representation $\rho_1 \times \cdots \times \rho_k \rtimes \tau_0$ has a unique irreducible subrepresentation $L(\rho_1 \otimes \cdots \otimes \rho_k \otimes \tau_0)$.*
(2) *Suppose that $m = 0$, $n_k > 1$ and $e(\rho_1) < \cdots < e(\rho_k) < 0$. Then the representation $\rho_1 \times \cdots \times \rho_k \rtimes 1_0$ (resp. $\rho_1 \times \cdots \times \rho_{k-1} \times s(\rho_k \rtimes 1_0)$) has a unique irreducible subrepresentation $L(\rho_1 \otimes \cdots \otimes \rho_k \otimes 1_0)$ (resp. $L(\rho_1 \otimes \cdots \otimes \rho_{k-1} \otimes s(\rho_k \otimes 1_0))$). Further, $L(\rho_1 \otimes \cdots \otimes \rho_k \otimes 1_0) \not\cong L(\rho_1 \otimes \cdots \otimes \rho_{k-1} \otimes s(\rho_k \otimes 1_0))$.*
(3) *Suppose that $m = 0$, $n_k = 1$ and $e(\rho_1) < \cdots < e(\rho_{k-1}) < -|e(\rho_k)| < 0$. Then the representation $\rho_1 \times \cdots \times \rho_k \rtimes 1_0$ has a unique irreducible subrepresentation $L(\rho_1 \otimes \cdots \otimes \rho_k \otimes 1_0)$.*

(ii) Let σ_0 be an irreducible admissible representation of $SO(2n, F)$. Then there exists a unique datum as in (i) such that $\sigma_0 \cong L(\cdot)$.

One remark is in order. We note that $SO(2, F) \cong F^\times$. Thus, a unitary character of $F^\times$ may be viewed as a tempered representation of $SO(2, F)$. This allows us to have the < 0 in part 3; if $e(\rho_k) = 0$, it is covered by part 1.

Now, we follow Theorem 6.2 to obtain the Langlands classification for $O(2n, F)$. First, we have to determine the set X_C. Since $\alpha_{n-1} < \alpha_n$ in the order on Π, we easily see that

$$\Phi \in X_C \iff (\alpha_{n-1} \in \Phi \quad \text{or} \quad \alpha_{n-1} \notin \Phi, \alpha_n \notin \Phi).$$

A.1) If $\alpha_{n-1} \in \Phi$, $\alpha_n \in \Phi$, then $C(\Phi) = \{1, s\}$. For every $x \in \mathfrak{a}^*_-$, we have $s \cdot x = x$, so $\mathfrak{a}^*_-(C) = \mathfrak{a}^*_-$ and $C(\Phi, x) = \{1, s\}$.
A.2) If $\alpha_{n-1} \in \Phi$, $\alpha_n \notin \Phi$, then $C(\Phi) = \{1\}$. It follows that $\mathfrak{a}^*_-(C) = \mathfrak{a}^*_-$ and $C(\Phi, x) = \{1\}$, for every $x \in \mathfrak{a}^*_-(C)$.
A.3) If $\alpha_{n-1} \notin \Phi$, $\alpha_n \notin \Phi$, then $C(\Phi) = \{1, s\}$. Take $x \in \mathfrak{a}^*_-$. Then $x = x_1e_1 + \cdots + x_ke_k$, where $x_1 < \cdots < x_{k-1} < -|x_k|$. The action of s on x is given by

$$s \cdot x = x_1e_1 + \cdots + x_{k-1}e_{k-1} - x_ke_k.$$

The condition $x \succeq s \cdot x$ implies $\langle x, \alpha_{k-1} \rangle \geq \langle x, \alpha_k \rangle$. This gives $x_{k-1} - x_k \geq x_{k-1} + x_k$, so $x_k \leq 0$. It follows that

$$\mathfrak{a}^*_-(C) = \{x_1e_1 + \cdots + x_ke_k \mid x_1 < \cdots < x_k \leq 0\}.$$

If $x_k < 0$, then $C(\Phi, x) = \{1\}$. If $x_k = 0$, then $C(\Phi, x) = \{1, s\}$.

Proposition 6.4 (The Langlands classification for $O(2n, F)$)**.** *(i) Let ρ_i, $i = 1, \dots, k$, be an irreducible essentially tempered representation of $GL(n_i, F)$ and τ an irreducible tempered representation of $O(2m, F)$. Suppose that $e(\rho_1) < \cdots < e(\rho_k) < 0$. Then the representation $\rho_1 \times \cdots \times \rho_k \rtimes \tau$ has a unique irreducible subrepresentation $L(\rho_1 \otimes \cdots \otimes \rho_k \otimes \tau)$.*

(ii) Let σ be an irreducible admissible representation of $O(2n, F)$. Then there exists a unique datum as in (i) such that $\sigma \cong L(\cdot)$.

Remark 6.5. (1) In section 2, a minor variation on Propositions 6.3 and 6.4 is used: Instead of having $\rho_1, \ldots, \rho_k$ essentially tempered with $\epsilon(\rho_1) < \cdots < \epsilon(\rho_k) < 0$ (resp., $\epsilon(\rho_1) < \cdots < \epsilon(\rho_{k-1}) < -|\epsilon(\rho_k)| < 0$ for Proposition 6.3 (3), we take $\rho_1, \ldots, \rho_k$ essentially square-integrable with $\epsilon(\rho_1) \leq \cdots \leq \epsilon(\rho_k) < 0$ (resp., $\epsilon(\rho_1) \leq \cdots \leq \epsilon(\rho_{k-1}) \leq -|\epsilon(\rho_k)| < 0$). This is justified by the following: If $\delta_1, \ldots, \delta_m$ are irreducible, square-integrable representations of $GL(n_1, F), \ldots GL(n_m, F)$, then $\delta_1 \times \cdots \times \delta_m$ is an irreducible, tempered representation. Further, any irreducible, tempered representation of $GL(n, F)$ can be written this way.

(2) We also need multiplicity one in the Langlands classification, i.e., that the Langlands subrepresentation appears with multiplicity one in the corresponding induced representation. We refer the reader to [B-W] in the connected case and [B-J2] in the non-connected case. (We remark that for $O(2n, F)$, multiplicity one may also be shown directly using the arguments of Lemma 3.4, [Jan4].)

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